

Semantical Analysis of Weak Kleene Logics

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Abstract

This paper presents a semantical analysis of the Weak Kleene Logics K_3^w and PWK from the tradition of Bochvar and Halldén. These are three-valued logics in which a formula takes the third value if *at least* one of its components does. The paper establishes two main results: a *characterization result* for the relation of logical consequence in PWK—that is, we individuate *necessary* and *sufficient* conditions for a set Δ of formulas to follow from a set Γ in PWK—and a *characterization result* for logical consequence in K_3^w . The paper also investigates two subsystems of K_3^w and PWK and discusses the relevance of the results against existing background. Finally, the paper discusses some issues related to Weak Kleene Logics—in particular, their philosophical interpretation and the reading of conjunction and disjunction—and points at some open issues.

Keywords: Weak Kleene tables, contamination, Bochvar, Halldén, variable inclusion requirements, containment logic.

1 Introduction

Weak Kleene logics are interpreted on the *weak tables* that are independently introduced by (Bochvar, 1938; Halldén, 1949; Kleene, 1952) in order to deal with one non-classical truth value beside the classical ones. The distinctive features of these logics is that, if A is assigned the third value, any formula B where A occurs will have the third value, regardless of the syntactic structure of B . This feature is called *contamination* (Ciuni, 2015; Ciuni and Carrara, 2016; Correia, 2002) or *infectiousness* (Ferguson, 2014b; Omori and Szmuc, 2017; Szmuc, 2017)—for clear reasons, we believe. Despite the variety of their philosophical applications (see below), there is little formal analysis of Weak Kleene logics. In particular, the relations of logical consequence in Weak Kleene logics remain largely unexplored.

This paper contributes to fill this gap and makes first steps toward a systematic analysis of Weak Kleene logics (from now on, WKLs). We focus on the two basic formalisms of the family: the so-called *internal* logic K_3^w by (Bochvar, 1938) and *Paraconsistent Weak Kleene* PWK investigated in (Ciuni, 2015) and (Ciuni and Carrara, 2016)—this is in turn the standard propositional fragment of the logic from (Halldén, 1949).¹ The main results of the paper are two *characterization results* for logical consequence in PWK and K_3^w , respectively—that is, we individuate *necessary* and *sufficient* conditions for a set Δ of formulas to be a PWK- (a K_3^w -) consequence of a set Γ of premises.

¹In this paper we include LP and PWK in the family of Kleene logics, since they are based of Kleene’s strong and weak tables, respectively. Notice, however, that the philosophical project by Kleene went along with paracomplete logics only. By including LP and PWK in the family of Kleene logics, we do not presuppose any historical filiation from Kleene’s project.

Besides, we prove that our main results can be generalized to the logics $L_{\mathbf{cb}'}$ and $L_{\mathbf{b'e}}$, which are two sublogics of PWK and K_3^w that have been recently introduced by (Barrio et al., 2016) and (Szmuc, 2017). We believe that this proves the fruitfulness of our methods and the possibility to extend it beyond the two three-valued WKLs. Finally, the paper briefly reviews some philosophical and technical points concerning WKLs, such as the interpretation of these logics, their extension with the so-called *meaningfulness operators*, and the addition of detachable conditionals.

The paper proceeds as follows. The remainder of this section introduces some background and discusses the relevance of the results and the methodology of the paper. Section 2 introduces the basic semantics of WKLs. Sections 3 and 4 establish the *characterization result* for PWK and K_3^w , respectively. The two sections also discuss some interesting features of the two theorems, and their relation with background literature. Section 5 presents the characterization results for $L_{\mathbf{cb}'}$ and $L_{\mathbf{b'e}}$. Section 6 reviews philosophical and technical points that are independent from the results of the paper, and yet help get a better grasp of WKLs. Any reader that has an expertise in WKLs can skip this section,² but we believe that the non-expert could gain a more complete understanding of WKLs by reading it. Section 7 opens some issues for future research, and Section 8 sums up the content of the paper.

1.1 Background

The so-called *weak tables* for the connectives (see Table 1) have been independently introduced by (Bochvar, 1938), (Halldén, 1949) and (Kleene, 1952). The name ‘weak tables’ is due to (Kleene, 1952),³ that defined them along the more famous *strong tables* below:

	$\neg A$	$A \vee B$	t	n	f	$A \wedge B$	t	n	f	$A \supset B$	t	n	f
t	f	t	t	t	t	t	t	n	f	t	t	n	f
n	n	n	t	n	n	n	n	n	f	n	t	n	n
f	t	f	t	n	f	f	f	f	f	f	t	t	t

(Kleene, 1952) just uses the weak tables in order to prove some facts on the definability of partial recursive shemata out of primitive recursive ones, but he finds that the strong tables are a more natural way to capture how partial information affects reasoning.⁴ By contrast, the weak tables lie at the hearth of the philosophical projects by (Bochvar, 1938) and (Halldén, 1949), that wish to capture the import of meaningless expressions on our reasoning (see Section 6).

The systems K_3^w and PWK—that we consider in this paper—are the so-called *internal logics* of the two projects.⁵ These logics capture the effects of meaninglessness on our reasoning, but they cannot express *that* a given statement is meaningless. The so-called *external logics* B_3 and H_3 provide this additional expressive power (see Sections 6 and 7).

²With the exception, maybe, of §6.4, which discusses very recent work on WKLs and is relevant for a better understanding of the last two paragraphs of Section 7.

³Since (Bochvar, 1938) and (Halldén, 1949) were not much known at the time, the tables retained the name given by Kleene, and the resulting logics were named after him.

⁴The strong tables provide the semantical basics of two well-known many-valued logics, namely the *paracomplete* K_3 from (Kleene, 1938), and its paraconsistent counterpart, the Logic of Paradox LP from (Priest, 2006). A slight adaptation of the tables as to include a fourth value is used to interpret the logic FDE by (Belnap, 1977).

⁵Notice that (Bochvar, 1938) calls Σ_0 what we call K_3^w today, and (Halldén, 1949) calls C_0 what we call PWK.

The weak Kleene tables have further echo in philosophy, where the idea of a *contaminating* non-classical value is recurrent: (Bochvar, 1938) applies them to *Russell’s paradox*, (Halldén, 1949) applies them to *vagueness*, *semantic paradoxes* and many other phenomena, (Prior, 1967) applies them to *reference to non-existent objects*. Besides, the five-valued logic independently designed by (Daniels, 1990) and (Priest, 2010)—on *fiction* and the *tetralemma* from Buddhist logic, respectively—includes a contaminating truth value. We expand on this in Section 6.

1.2 Relevance and Methodolgy of the Paper

There is increasing interest today toward WKLs. (Coniglio and Corbalan, 2012) develop *sequent calculi* for them, (Ciuni, 2015) explores some connections between PWK and LP, (Ciuni and Carrara, 2016) focus on logical consequence in PWK. Also, (Barrio et al., 2016), (Cobreros and Carrara, 2016) (in progress), and (Bonzio et al., 2017), (Fitting, 2006), (Omori and Szmuc, 2017) show that there are efforts converging on WKLs from different angles (proof-theory, algebra, philosophical logic). Finally, logics related to WKLs are investigated in (Ferguson, 2014a; Ferguson, 2014b; Szmuc, 2017).

Relevance of the results. The relevance of Theorem 3.4 and Theorem 4.3 (Sections 3 and 4, respectively) follows from the *understanding* of WKLs that they secure, and the *progress* they make w.r.t (with respect to) *existing background*. For instance, Theorem 3.4 gives a unified explanation of the failures in PWK, including failure of *Conjunctive Simplification*, that is the most distinctive feature of PWK. Similarly, Theorem 4.3 explains all the failures in K_3^w , and most prominently failure of *Disjunctive Additivity*.

As for connections with existing background, the results from (Paoli, 2007), (Coniglio and Corbalan, 2012), and (Urquahrt, 2002) all follow from our theorems. The first two results can be derived from Theorem 3.4, the last one can be derived from Theorem 4.3. In particular, (Paoli, 2007) establishes a characterization result for the FDE-fragment of PWK. Also, (Coniglio and Corbalan, 2012) discuss one direction of Theorem 3.4, but their results cannot be directly generalized to a characterization result. Also, notice that Theorem 3.4 generalizes the main result from (Ciuni and Carrara, 2016). Finally, (Urquahrt, 2002) captures the single-conclusion version of logical consequence in K_3^w , and his result immediately follows from our results. A further progress w.r.t. (Ciuni and Carrara, 2016) is that the latter does not provide any characterization result for K_3^w .

Methodology of the paper. The paper deploys a semantical methodology. In particular, we will focus on the interpretation of the connectives on Table 1, which implicitly defines the so-called Weak Kleene algebra, and on the different options for designation of truth values and definition of logical consequence. This approach is overall equivalent to the investigation of WKLs as matrix-based logics. We do not go through proof-theoretical results for WKLs in this paper, since well-understood proof systems for them have been established already. In particular, (Bonzio et al., 2017) provide a complete Hilbert-style proof-system for PWK, (Priest, 2019) presents a complete natural deduction system for PWK, and (Coniglio and Corbalan, 2012) devise complete sequent calculi for K_3^w and PWK.

2 Weak Kleene Logics: Language and Semantics

The language $\mathcal{L}$ of WKLs is the standard propositional language. Given a denumerable set $\text{Var} = \{p, q, r, \dots\}$ of propositional variables, the language is defined by the following Backus-Naur Form (BNF):

$$\Phi_{\mathcal{L}} ::= p \mid \neg A \mid A \vee B \mid A \wedge B \mid A \supset B$$

We use $\Gamma, \Phi, \Psi, \Sigma, \dots$ to denote *sets* of arbitrary formulas. A useful notation is this: $\text{Var}(\Gamma)$ is the set of variables occurring in some formula $A \in \Gamma$. Whenever $\Delta = \{A\}$, we write $\text{Var}(A)$ instead of $\text{Var}(\{A\})$.

Propositional variables from Var are interpreted by a valuation function $V : \text{Var} \rightarrow \{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ that assigns one out of three values to each $p \in \text{Var}$. The valuation extends to arbitrary formulas according to the following definition:

Definition 2.1 (Valuation) A valuation $V : \Phi_{\mathcal{L}} \rightarrow \{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ is the unique extension of a mapping $V : \text{Var} \rightarrow \{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ that is induced by the tables from Table 1.

Table 1: Weak Tables for Logical Connectives in $\mathcal{L}$

	$\neg A$	$A \vee B$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$	$A \wedge B$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$	$A \supset B$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{t}$

We let $\mathcal{V}$ be the set $V, V', V'', \dots$ of valuations conforming to Definition 2.1. We call $\mathcal{V}_{\text{CL}}$ the set of valuations of Classical Logic CL, with $\mathcal{V}_{\text{CL}} = \{V \in \mathcal{V} \mid V(p) \neq \mathbf{n} \text{ for every } p \in \text{Var}\}$. It is clear that $\mathcal{V}_{\text{CL}} \subset \mathcal{V}$.

Table 1 provides the *weak tables* from (Kleene, 1952, §64), that obtain ‘by supplying [the third value] throughout the row and column headed by [the third value]’. The way $\mathbf{n}$ transmits is usually called *contamination* (or *infection*), since the value propagates from any $A \in \Phi_{\mathcal{L}}$ to any construction $k(A, B)$, independently from the value of B (here, k is any connective defined in terms of $\vee, \wedge, \supset$). The following gives a straightforward and intuitive expression to *contamination*:

Fact 2.1 (Contamination) For all formulas A in $\Phi_{\mathcal{L}}$ and valuation $V \in \mathcal{V}$:

$$V(A) = \mathbf{n} \text{ iff } V(p) = \mathbf{n} \text{ for some } p \in \text{Var}(A)$$

The LTR (left-to-right) direction is shared by all the most widespread three-valued logics; the RTL (right-to-left) direction is clear from Table 1, and it implies that A takes value $\mathbf{n}$ if some $p \in \text{Var}(A)$ has the value, and *no matter* what the value of q is for any $q \in \text{Var}(A) \setminus \{p\}$.

3 Characterizing Logical Consequence in PWK

The *Paraconsistent Weak Kleene* logic PWK originates from (Halldén, 1949), and it is the standard propositional fragment of the *logic of nonsense* presented there. Halldén believes that logic should

also account for *nonsensical sentences*. In his jargon, these are expressions that are syntactically well-formed and yet fail to convey a proposition. In this category, (Halldén, 1949) includes the Liar and Russell’s paradox as well as ambiguous and vague sentences, and sentences referring to non-existent objects. Halldén’s treatment can be captured semantically by (1) interpreting *nonsense* (or *nonsensical*) as the non-classical truth value satisfying the tables from Table 1, (2) accepting the very same tables for $\neg$, $\vee$, $\wedge$, $\supset$, and (3) designating **n** alongside **t**. Halldén’s logic of nonsense H_3 also includes a unary connective that allows to build statements such as ‘ A is meaningful’ and does not satisfy contamination. PWK corresponds to the so-called *internal logic* developed by (Halldén, 1949)—that is, PWK includes just standard propositional connectives and cannot express that its own statements are meaningless (or meaningful).⁶

The above remarks yields the following definition of logical consequence in PWK as preservation of **t** or **n**. From now on, we will call this ‘PWK-consequence’:

Definition 3.1 (PWK-Consequence) PWK-consequence is a relation $\models_{\text{PWK}} \subseteq \wp(\Phi_{\mathcal{L}}) \times \wp(\Phi_{\mathcal{L}})$ such that:

$\Gamma \models_{\text{PWK}} \Delta$ iff For every $V \in \mathcal{V}$, if $V(A) \in \{\mathbf{t}, \mathbf{n}\}$ for every $A \in \Gamma$, then $V(B) \in \{\mathbf{t}, \mathbf{n}\}$ for some $B \in \Delta$.

If $\Gamma \models_{\text{PWK}} \Delta$, then we say that Δ is a PWK-consequence of Γ , or equivalently, Δ follows from Γ in PWK. We write $A, B \models_{\text{PWK}} C, D$ for $\{A, B\} \models_{\text{PWK}} \{C, D\}$. A set $\Delta \subseteq \wp(\Phi_{\mathcal{L}})$ is a *tautology* iff $\emptyset \models_{\text{PWK}} \Delta$. We define the set $\mathcal{V}_{\text{PWK}}(\Gamma) = \{V \in \mathcal{V} \mid V(A) \in \{\mathbf{t}, \mathbf{n}\} \text{ for every } A \in \Gamma\}$ of the valuations in $\mathcal{V}$ that *satisfy* a set Γ of formulas in PWK. Γ is satisfiable in PWK iff $\mathcal{V}_{\text{PWK}}(\Gamma) \neq \emptyset$, and it is a tautology iff $\mathcal{V}_{\text{PWK}}(\Gamma) = \mathcal{V}$.

We establish some preliminary results and then we go to the characterization result for PWK (Theorem 3.4).

Fact 3.1 If $\Gamma \models_{\text{PWK}} \Delta$, then $\Gamma \models_{\text{CL}} \Delta$

Due to $\mathcal{V}_{\text{CL}} \subset \mathcal{V}$. The other direction clearly does not hold, since , by the definition of $\mathcal{V}_{\text{CL}}$, $V \notin \mathcal{V}_{\text{CL}}$ for all the $V \in \mathcal{V}$ such that $V(p) = \mathbf{n}$ for at least a $p \in \text{Var}$.

Fact 3.2 $\emptyset \models_{\text{PWK}} \Delta$ iff $\emptyset \models_{\text{CL}} \Delta$

The LTR direction is a special case of Fact 3.1. As for RTL direction, suppose it fails. Then there must be a valuation $V \in \mathcal{V}$ such that $V(B) = \mathbf{f}$ for every $B \in \Delta$. By Fact 2.1, $V(p) \neq \mathbf{n}$ for every $p \in \text{Var}(\Delta)$, and hence V could be extended to a classical valuation $V' \in \mathcal{V}_{\text{CL}}$ such that $V'(p) = V(p)$ if $p \in \text{Var}(\Delta)$. This implies $V'(B) = \mathbf{f}$ for every $B \in \Delta$, contrary to the assumption that $\emptyset \models_{\text{CL}} \Delta$. The following will also be useful when proving Theorem 3.4:

Fact 3.3 (Monotonicity) If $\Gamma \models_{\text{PWK}} \Delta$ then $\Gamma, \Sigma \models_{\text{PWK}} \Delta$.

This follows from $\mathcal{V}_{\text{PWK}}(\Gamma \cup \Sigma) \subseteq \mathcal{V}_{\text{PWK}}(\Gamma)$. Given a pair $\{\Gamma, \Delta\}$ from $\wp(\Phi_{\mathcal{L}}) \times \wp(\Phi_{\mathcal{L}})$, we define the set

$$\mathbf{G}_{\Gamma, \Delta} = \{\Gamma' \subseteq \Gamma \mid \Gamma' \neq \emptyset \text{ and } \text{Var}(\Gamma') \subseteq \text{Var}(\Delta)\}$$

of those non-empty subsets of Γ whose variables are all included in the variables from Δ . We will need this notion in the following results.

⁶We come back to Halldén’s external logic and his interpretation of the weak Kleene machinery in Section 6.

Proposition 3.2 *If $\Gamma \models_{\text{PWK}} \Delta$ and $\emptyset \not\models_{\text{PWK}} \Delta$, then $\mathbf{G}_{\Gamma, \Delta} \neq \emptyset$.*

Proof. Assume the antecedent ($\Gamma \models_{\text{PWK}} \Delta$ and $\emptyset \not\models_{\text{PWK}} \Delta$) as an initial hypothesis, and suppose $\mathbf{G}_{\Gamma, \Delta} = \emptyset$. The latter implies that for every $A \in \Gamma$, $\text{Var}(A) \not\subseteq \text{Var}(\Delta)$. For every $A \in \Gamma$, take the set $D_A = \text{Var}(A) \setminus \text{Var}(\Delta)$. Since $\text{Var}(A) \not\subseteq \text{Var}(\Delta)$, we have $D_A \neq \emptyset$. Now take a valuation $V \in \mathcal{V}$ such that

- $V(B) = \mathbf{f}$ for every $B \in \Delta$,
- $V(p) = \mathbf{n}$ for every $p \in D_A$ and $A \in \Gamma$.

This valuation exists, since the value-assignments to every element of D_A and $\text{Var}(\Delta)$ are independent. We have that $V(B) = \mathbf{f}$ for every $B \in \Delta$ and, by Fact 2.1, $V(A) = \mathbf{n}$ for every $A \in \Gamma$. As a consequence, $\Gamma \not\models_{\text{PWK}} \Delta$. But this contradicts the initial hypothesis. ■

The statement in Proposition 3.2 is equivalent with: (i) if $\Gamma \models_{\text{PWK}} \Delta$ and $\mathbf{G}_{\Gamma, \Delta} = \emptyset$, then $\emptyset \models_{\text{PWK}} \Delta$, and (ii) if $\emptyset \not\models_{\text{PWK}} \Delta$ and $\mathbf{G}_{\Gamma, \Delta} = \emptyset$, then $\Gamma \not\models_{\text{PWK}} \Delta$. The only case where the (sets of) variables of premises and conclusions are disjoint in a PWK-valid inference is the case where the conclusion is a tautology.

Proposition 3.3 *If $\Gamma \models_{\text{PWK}} \Delta$ and $\emptyset \not\models_{\text{PWK}} \Delta$, then $\exists \Gamma' \in \mathbf{G}_{\Gamma, \Delta}$ such that $\Gamma' \models_{\text{CL}} \Delta$.*

Proof. We prove that the antecedent of the statement implies $\exists \Gamma' \in \mathbf{G}_{\Gamma, \Delta}$ such that $\Gamma' \models_{\text{PWK}} \Delta$. From this and Fact 3.1, the result follows. Assume $\emptyset \not\models_{\text{PWK}} \Delta$ and $\Gamma \models_{\text{PWK}} \Delta$ as the initial hypothesis. Suppose that $\Gamma' \not\models_{\text{PWK}} \Delta$ for every $\Gamma' \in \mathbf{G}_{\Gamma, \Delta}$. This implies that there is a valuation $V \in \mathcal{V}$ such that $V(B) = \mathbf{f}$ for every $B \in \Delta$ and $V(A) \in \mathcal{D}_{\text{PWK}}$ for every $A \in \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$. Due to Fact 2.1, we have $V(p) \neq \mathbf{n}$ for every $p \in \text{var}(\Delta)$ and, by the definition of $\mathbf{G}_{\Gamma, \Delta}$, $V(A) \neq \mathbf{n}$ for every $A \in \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$. This implies $V(A) = \mathbf{t}$ for every $A \in \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$. V can be extended to a valuation $V' \in \mathcal{V}$ such that $V'(p) = V(p)$ if $p \in \text{var}(\Delta)$, and $V'(p) = \mathbf{n}$ otherwise. This implies $V'(C) = \mathbf{n}$ for every $C \in \Gamma$ such that $C \notin \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$, $V'(A) = \mathbf{t}$ for every $A \in \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$, and $V'(B) = \mathbf{f}$ for every $B \in \Delta$. But this contradicts the initial hypothesis. From this, we have that, if $\Gamma \models_{\text{PWK}} \Delta$ and $\emptyset \not\models \Delta$, then $\exists \Gamma' \in \mathbf{G}_{\Gamma, \Delta}$ such that $\Gamma' \models_{\text{PWK}} \Delta$. From this and Fact 3.1, we conclude that, if $\Gamma \models_{\text{PWK}} \Delta$ and $\emptyset \not\models \Delta$, then $\exists \Gamma' \in \mathbf{G}_{\Gamma, \Delta}$ such that $\Gamma' \models_{\text{CL}} \Delta$. ■

Proposition 3.2 states that, unless Δ is a tautology, if the inference from Γ to Δ is valid, then there must be some non-empty subset Γ' of Γ whose variables are all contained in the variables of Δ . Proposition 3.3 establishes the additional requirement that Δ classically follows from (at least) one such subset Γ' .

Just to give a concrete grasp of this, consider that $A, (B \vee C), D \models_{\text{PWK}} (B \vee C) \wedge D$. Of course, the variables from the premise-set need not be a subset of those from the conclusion (suppose A, B, C, D are four distinct propositional variables p, q, r, s). However, there is a subset of the premise-set (namely, $\{(B \vee C), D\}$) that implies the conclusion in CL, and whose variables are all contained in the conclusion. Now we are ready to present the main result of this section:

Theorem 3.4

$$\Gamma \models_{\text{PWK}} \Delta \text{ iff } \begin{cases} \emptyset \models_{\text{CL}} \Delta, & \text{or} \\ \Gamma' \models_{\text{CL}} \Delta & \text{for at least a non-empty} \\ & \Gamma' \subseteq \Gamma \text{ s.t. } \text{Var}(\Gamma') \subseteq \text{Var}(\Delta). \end{cases}$$

Proof. The LTR direction immediately follows from Fact 3.1, Fact 3.2 and Proposition 3.3. As for the RTL direction, we prove it in two steps. If $\emptyset \models_{\text{CL}} \Delta$, we have $\Gamma \models_{\text{PWK}} \Delta$ by Fact 3.2 and Fact 3.3. Let us now assume $\emptyset \not\models_{\text{CL}} \Delta$ and $\Gamma' \models_{\text{CL}} \Delta$ for at least a non-empty $\Gamma' \subseteq \Gamma$ s.t. $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$. Let $\overline{\mathcal{V}_{\text{CL}}}(\Delta) = \{V \in \mathcal{V}_{\text{CL}} \mid V(B) = \mathbf{f} \text{ for every } B \in \Delta\}$ be the set of ‘classical’ valuations in $\mathcal{V}$ that give truth value $\mathbf{f}$ to every $B \in \Delta$ —and, of course, $\mathbf{n}$ to no variable. The first assumption implies that $\overline{\mathcal{V}_{\text{CL}}}(\Delta) \neq \emptyset$. The second implies $\mathcal{V}_{\text{CL}}(\Gamma') \subseteq \mathcal{V}_{\text{CL}}(\Delta)$. To establish $\Gamma' \models_{\text{PWK}} \Delta$, fix any valuation $V \in \mathcal{V}$ such that $V(A) \in \{\mathbf{n}, \mathbf{t}\}$ for every $A \in \Gamma'$. Our goal is to show that $V(B) = \{\mathbf{n}, \mathbf{t}\}$ for some $B \in \Delta$. We consider two cases:

Case 1): $V(A) = \mathbf{n}$ for some $A \in \Gamma'$. Fix some formula $C \in \Gamma'$ such that $V(C) = \mathbf{n}$. By Fact 2.1, there is a $q \in \text{var}(C)$ such that $V(q) = \mathbf{n}$. Since $\text{var}(\Gamma') \subseteq \text{var}(\Delta)$ and $C \in \Gamma'$, $q \in \text{var}(\Delta)$. Again by Fact 2.1, this implies $V(B) = \mathbf{n}$ for some $B \in \Delta$, as desired.

Case 2): $V(A) \neq \mathbf{n}$ for every $A \in \Gamma'$. This implies that $V(A) = \mathbf{t}$ for every $A \in \Gamma'$. Since $\Gamma' \models_{\text{CL}} \Delta$ (by our initial hypothesis), we have that $V(B) = \mathbf{t}$ for some $B \in \Delta$, as desired.

Since these two cases are jointly exhaustive (that is, every valuation $V \in \mathcal{V}(\Gamma')$ fits in one of them), we conclude $\Gamma' \models_{\text{PWK}} \Delta$. From this and the monotonicity of $\models_{\text{PWK}}$ (Fact 3.3), it follows that $\Gamma \models_{\text{PWK}} \Delta$. ■

An immediate corollary of Theorem 3.4 is the following:

Corollary 3.5 *If $\text{Var}(\Gamma) \subseteq \text{Var}(\Delta)$ and $\Gamma \models_{\text{CL}} \Delta$, then $\Gamma \models_{\text{PWK}} \Delta$.*

That is: $\text{Var}(\Gamma) \subseteq \text{Var}(\Delta)$ is *sufficient* for a classical inference to be PWK-valid. This is just a special case of the RTL direction of Theorem 3.4—namely the case where $\Gamma \in \mathbf{G}_{\Gamma, \Delta}$.

Remark 3.6 Of course, the general inclusion requirement ‘ $\text{Var}(\Gamma) \subseteq \text{Var}(\Delta)$ and $\Gamma \models_{\text{CL}} \Delta$ ’ from Corollary 3.5 *cannot* provide a *necessary* condition for a classical inference to be PWK-valid, even in case Δ is not a tautology. Monotonicity of $\models_{\text{PWK}}$ (Fact 3.3) suffices to see this: if $\Gamma \models_{\text{PWK}} \Delta$, then $\Psi \cup \Gamma \models_{\text{PWK}} \Delta$, even if $\text{Var}(\Psi) \not\subseteq \text{Var}(\Delta)$. However, the general inclusion requirement $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$ —since now on, GIR—still proves illuminating in understanding the insight of Theorem 3.4. Indeed, given a classically valid inference from Γ to Δ , with Δ not a classical tautology, the strategy prescribed by Theorem 3.4 is to find a *subset* $\Gamma' \subseteq \Gamma$ such that the pair (Γ', Δ) satisfies GIR. If there is at least one such set, then $\Gamma \models_{\text{PWK}} \Delta$, otherwise, $\Gamma \not\models_{\text{PWK}} \Delta$. Ideally, in case $\Gamma \notin \mathbf{G}_{\Gamma, \Delta}$, the theorem suggests that we find the *smallest subsets* that, together with Δ , satisfies GIR.

3.1 Examples of classical failures

We can use Theorem 3.4 in order to individuate interesting failures of classically valid inferences in PWK. For instance, *Ex Contradictione Quodlibet* (ECQ) and *Conjunctive Simplification* (CS) fail: these are inferences where the variables from the premise need not to be included in those from the conclusion, to the effect that the conditions from Theorem 3.4 is not met.

$$\begin{array}{ll} A \wedge \neg A \not\models_{\text{PWK}} B & \text{ECQ} \\ A \wedge B \not\models_{\text{PWK}} B & \text{CS} \end{array}$$

Failure of ECQ implies that PWK is *paraconsistent*—whence the name.⁷ Notice that, in the failure of CS, Fact 2.1 is crucial: $V(A) = \mathbf{n}$ suffices to have $V(A \wedge B) = \mathbf{n}$. Notice that, contrary to PWK,

⁷Any valuation $V \in \mathcal{V}$ with $V(A) = \mathbf{n}$ and $V(B) = \mathbf{f}$ suffices to invalidate the rule.

the Logic of Paradox LP from (Priest, 2006) verifies CS. Other examples include:

$A, A \supset B \not\models_{\text{PWK}} B$	<i>Modus Ponens</i>	MP
$\neg B, A \supset B \not\models_{\text{PWK}} \neg A$	<i>Modus Tollens</i>	MT
$A \supset B, B \supset C \not\models_{\text{PWK}} A \supset C$	<i>Transitivity of Conditional</i>	TR $\supset$
$A \supset (B \wedge \neg B) \not\models_{\text{PWK}} \neg A$	<i>Reductio ad Absurdum</i>	RAA

None of these satisfy the conditions from Theorem 3.4.⁸ The theorem also helps individuate those classical inferences that are valid in PWK. We just give two examples:

$$\begin{array}{ll} A \wedge B \models_{\text{PWK}} A, B & \text{CS}' \\ A \wedge \neg A \models_{\text{PWK}} A \wedge B & \end{array}$$

Both rules satisfy the conditions from Theorem 3.4: there is a subset of the premise-set (in this case, the premise-set itself) such that you can classically infer the conclusion-set from it, and such that its variables are included in those from the premise-set.⁹

3.2 Discussion of Theorem 3.4

Theorem 3.4 is an advancement w.r.t. existing results on PWK-consequence: relevant theorems by (Paoli, 2007) and (Coniglio and Corbalan, 2012) turn to be consequences of Theorem 3.4 and Corollary 3.5, respectively. Below, we discuss the connection between the results from (Coniglio and Corbalan, 2012) and (Paoli, 2007) and our ones, and we explain why our results make a genuine progress w.r.t. theirs.

The characterization result by (Paoli, 2007) concerns the FDE-fragment H of PWK. This is PWK augmented with the (standardly defined) entailment connective $\Rightarrow$ from the FDE-tradition. In particular, for every $A, B \in \Phi_{\mathcal{L}}$, we have: $(\star) \models_{\text{H}} A \Rightarrow B$ iff $A \models_{\text{PWK}} B$.¹⁰ Paoli proves:

Proposition 3.7 (Theorem 1 of (Paoli, 2007)) $\models_{\text{H}} A \Rightarrow B$ iff $A \models_{\text{CL}} B$ and either $\emptyset \models_{\text{CL}} B$ or $\text{Var}(A) \subseteq \text{Var}(B)$.

Given $\star$, Theorem 1 from (Paoli, 2007) follows as a special case of our Theorem 3.4, namely the case where $\Gamma = \{A\}$ and $\Delta = \{B\}$ for some $A, B \in \Phi_{\mathcal{L}}$. (Coniglio and Corbalan, 2012) define a *sequent calculi* for K_3^w and PWK and present some results on the relation between PWK-consequence and CL-consequence. In particular, they prove:

Proposition 3.8 (Theorem 8 of (Coniglio and Corbalan, 2012)) If $\Gamma \models_{\text{CL}} B$ and $(\text{Var}(\Gamma) \subseteq \text{Var}(B) \text{ or } \emptyset \models_{\text{CL}} B)$, then $\Gamma \models_{\text{PWK}} B$.

which is in turn a single-conclusion version of Corollary 3.5. More precisely, Corollary 3.5 together with Fact 3.2 and Fact 3.3 imply Corollary 3.8. As for Corollary 3.5, the theorem from

⁸Once again, it is easy to check these failures via counterexample. Any $V \in \mathcal{V}$ such that $V(A) = \mathbf{n}$ and $V(B) = \mathbf{f}$ falsifies MP. $V(A) = \mathbf{t}$ and $V(B) = \mathbf{n}$, falsifies MT and RAA, $V(A) = \mathbf{t}$, $V(B) = \mathbf{n}$ and $V(C) = \mathbf{f}$ falsifies TR $\supset$.

⁹Fact 2.1 plays a crucial role in the second inference: $V(A) = \mathbf{n}$ suffices to have both $V(A \wedge \neg A) = \mathbf{n}$ and $V(A \wedge B) = \mathbf{n}$.

¹⁰There are many different ways to characterize FDE-logics and -fragments. Here, we find it natural to follow the one adopted in (Paoli, 2007).

(Coniglio and Corbalan, 2012) provides a *sufficient* condition for PWK-consequence, and yet it does not provide a *necessary* condition for it: we have seen that $A, B \vee C, D \models_{\text{PWK}} (B \vee C) \wedge D$, and clearly the variable inclusion condition from Proposition 3.8/Corollary 3.5 is not satisfied, since $\text{Var}(\{A, B \vee C, D\}) \not\subseteq \text{Var}(\{(B \vee C) \wedge D\})$. As a consequence, Proposition 3.8/Corollary 3.5 cannot be generalized to a characterization of PWK-consequence.

The result from (Paoli, 2007) leaves a question open: ‘How are we to generalize (Paoli, 2007, Theorem 1) beyond the single-premise/single-conclusion case?’. This in turn equates with asking ‘How are we to generalize (Coniglio and Corbalan, 2012, Theorem 8) to a characterization of PWK-consequence?’—where the latter is the main question left open by the theorem in (Coniglio and Corbalan, 2012). Theorem 3.4 provides an answer to both questions.

An interesting point is that it is not possible to understand the necessary conditions for PWK-consequence if we keep our focus on the limit-cases where $\text{Var}(\Gamma) \subseteq \text{Var}(\Delta)$, as (Paoli, 2007) and (Coniglio and Corbalan, 2012) do. What is at stake here is how information-related requirements interact with reasoning, and of course the kind of consequence we are considering makes a crucial difference. As for the single-premise/single-conclusion case on which (Paoli, 2007) focuses: if $\Gamma \models_{\text{CL}} \Delta$, and $\Gamma = \{A\}$, $\Delta = \{B\}$, then B follows from A if and only if $\mathbf{G}_{\Gamma, \Delta} = \{\Gamma\}$; hence the strategy prescribed by GIR and the strategy prescribed by Theorem 3.4 collapse on one another, and it is not possible to appreciate the different contributions they can make in the more general multiple-premise/multiple-conclusion cases. As for the multiple-premise/single-conclusion case, if $\Gamma \models_{\text{CL}} \Delta$ and $\Gamma = \{A_1, \dots, A_n\}$, $\Delta = \{B\}$, then satisfaction of GIR suffices for Δ to follow Γ in PWK, but as we have seen, GIR is by no means necessary—see Remark 3.6. But failure of GIR as a necessary condition leaves us, per se, without any hint on what condition we should look for. Thus, an exclusive focus on the limit cases where $\text{Var}(\Gamma) \subseteq \text{Var}(\Delta)$ may obscure the contribution that subsets of Γ can make, and the relevance of the condition $\Gamma' \in \mathbf{G}_{\Gamma, \Delta}$ for some Γ' such that $\Gamma' \models_{\text{PWK}} \Delta$.¹¹

Finally, Theorem 3.4 generalizes Theorem 3.8 from (Ciuni and Carrara, 2016), which provides a characterization result for single-conclusion PWK-consequence. In particular, Theorem 3.8 from (Ciuni and Carrara, 2016) is the special case of Theorem 3.4 where $\Delta = \{B\}$. Theorem 3.4 makes a progress w.r.t. (Ciuni and Carrara, 2016, Theorem 3.8) in that the former covers a more general case than the latter. Second, the proof of Theorem 3.4 is significantly different from (and in our view, more insightful than) the proof of Theorem 3.8 in (Ciuni and Carrara, 2016).¹² Also, we see below (Section 4) that Theorem 3.4 proves helpful in the characterization result for multiple-conclusion (and multiple-premise) $\mathbf{K}_3^w$ -consequence—Theorem 4.3 from Section 4. The latter cannot be as easily generalized from the characterization of single-conclusion $\mathbf{K}_3^w$ -consequence (Corollary 4.7 below). In this light Theorem 3.4 does not just offer an interesting generalization of an existing result, but it is also functional to the general characterization of $\mathbf{K}_3^w$ -consequence (see Section 4).

We close with a short sum-up of our discussion on variable-inclusion requirements. Consider

¹¹Notice that this considerations do not contradict Remark 3.6. There, we suggest that GIR can help familiarize with the variable-inclusion requirement from Theorem 3.4 works, since the former provides an easy limit-case of the latter. What we are denying in the above remark, by contrast, is that focus on situations where GIR is satisfied can provide a good *heuristics* in order to find the right necessary and sufficient conditions for PWK-consequence.

¹²We thank an anonymous reviewer for suggesting further changes and improvements in the proof of Theorem 3.4.

the following statements:

- (i) $\Gamma \models_{\text{PWK}} \Delta$;
- (ii) $\emptyset \models_{\text{CL}} \Delta$;
- (iii) $\Gamma' \models_{\text{CL}} \Delta$ for some $\Gamma' \subseteq \Gamma$ s.t. $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$.

The RTL direction of Theorem 3.4 implies that (ii) and (iii) are *individually sufficient* for (i) (each of them implies $\Gamma \models_{\text{PWK}} \Delta$). The LTR direction of Theorem 3.4 implies that (ii) and (iii) are *jointly necessary* for (i) (if neither condition holds, then $\Gamma \not\models_{\text{PWK}} \Delta$). Now consider

$$(iv) \quad \Gamma \models_{\text{CL}} \Delta \text{ and } \text{Var}(\Gamma) \subseteq \text{Var}(\Delta).$$

Since (iv) implies (iii), we have that (iv) is individually sufficient for (i). The fact that (ii) and (iv) are individually sufficient for (i) is in turn Corollary 3.5 or, equivalently, Theorem 8 by (Coniglio and Corbalan, 2012). However, (ii) and (iv) are not jointly necessary, since (i) may hold even if both conditions are false.

4 Characterizing Logical Consequence in K_3^w

Logic K_3^w is introduced by (Bochvar, 1938) in order to deal with *meaningless expressions*. These are basically Halldén’s nonsensical sentences (expressions that are syntactically well-formed and yet fail to convey a proposition), and (Bochvar, 1938) treats *meaninglessness* as a non-classical truth value obeying the tables from Table 1, exactly as Halldén. A further analogy is that also Bochvar’s machinery can be divided in two: the *internal logic* K_3^w , that includes the standard propositional connectives and cannot express the meaninglessness of its formulas, and the external logic B , that adds a unary connective in order to say that its formulas are (or aren’t) meaningful *and* true.

If we assume again the specific semantical reading from Section 3 (which just expresses the methodological angle of the paper), then we can represent the crucial difference between (Bochvar, 1938) and (Halldén, 1949) by saying that, in K_3^w , only classical truth must be preserved in reasoning. Beside, (Bochvar, 1938) is concerned with blocking Russell’s paradox in set theory, and, contrary to (Halldén, 1949), he devises no application to semantic paradoxes, vagueness, or denotational failures.¹³

The logic K_3^w obtains by defining logical consequence as preservation of **t**. From now on, we will call this ‘ K_3^w -consequence’:

Definition 4.1 (K_3^w -consequence) K_3^w -consequence is a relation $\models_{K_3^w} \subseteq \wp(\Phi_{\mathcal{L}}) \times \wp(\Phi_{\mathcal{L}})$ such that:

$$\Gamma \models_{K_3^w} \Delta \text{ iff For every } V \in \mathcal{V}, \text{ if } V(A) = \mathbf{t} \text{ for every } A \in \Gamma, \text{ then } V(B) = \mathbf{t} \text{ for some } B \in \Delta$$

If $\Gamma \models_{K_3^w} \Delta$, we say that Δ is a K_3^w -consequence of Γ or, equivalently, that Δ follows from Γ in K_3^w . The definitions of tautology, set of valuations satisfying a formula, and satisfiability are just as for PWK, with **t** replacing $\{\mathbf{t}, \mathbf{n}\}$. For every $\Gamma \subseteq \Phi_{\mathcal{L}}$, let $\Gamma^\neg$ be the set $\{\neg A \in \Phi_{\mathcal{L}} \mid A \in \Gamma\}$.

Before presenting the characterization result for K_3^w , we give an idea of the strategy of the proof. We characterize K_3^w -consequence (Theorem 4.3) by exploiting the expected *duality* between PWK and K_3^w (Proposition 4.2). Once this is done, Theorem 4.3 simply follows from the duality and

¹³We come back to Bochvar’s external logic and his interpretation of the weak Kleene machinery in Section 6.

Theorem 3.4. As for Proposition 4.2, this is not a particularly surprising or challenging result, but notice that the duality result is unprecedented and is a key to transfer the result from Theorem 3.4 to the case of K_3^w -consequence. Now for the duality result:

Proposition 4.2 $\Gamma \models_{K_3^w} \Delta$ iff $\Delta^\neg \models_{\text{PWK}} \Gamma^\neg$

Proof: $\Delta^\neg \models_{\text{PWK}} \Gamma^\neg$ holds iff for every valuation $V \in \mathcal{V}$, if $V(\neg B) = \{\mathbf{t}, \mathbf{n}\}$ for every $B \in \Delta$, then $V(\neg A) = \{\mathbf{t}, \mathbf{n}\}$ for some $A \in \Gamma$. This equates with having that, for every valuation $V \in \mathcal{V}$, if $V(\neg A) = \mathbf{f}$ for every $A \in \Gamma$, then $V(\neg B) = \mathbf{f}$ for some $B \in \Delta$. The latter is in turn equivalent with having that, if $V(A) = \mathbf{t}$ for every $A \in \Gamma$, then $V(B) = \mathbf{t}$ for some $B \in \Delta$. But this implies $\Gamma \models_{K_3^w} \Delta$. This suffices to prove also the RTL direction of the statement. ■

A characterization result for K_3^w immediately follows from Theorem 3.4 and Proposition 4.2:

Theorem 4.3

$$\Gamma \models_{K_3^w} \Delta \text{ iff } \begin{cases} \Gamma \models_{\text{CL}} \emptyset & \text{or} \\ \Gamma \models_{\text{CL}} \Delta' & \text{for at least a non-empty} \\ & \Delta' \subseteq \Delta \text{ s.t. } \text{Var}(\Delta') \subseteq \text{Var}(\Gamma). \end{cases}$$

Theorem 4.3 imposes the requirement that $\Gamma \models_{\text{CL}} \Delta'$ and $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$ for at least a non-empty $\Delta' \subseteq \Delta$ such that $\Gamma \models_{\text{CL}} \Delta'$. This in turn dualizes the inclusion requirement from Proposition 3.3 and Theorem 3.4. A corollary of Theorem 4.3 is:

Corollary 4.4 If $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$ and $\Gamma \models_{\text{CL}} \Delta$, then $\Gamma \models_{K_3^w} \Delta$.

which dualizes Corollary 3.5 and captures the special case where $\Delta \in \mathbf{G}_{\Delta, \Gamma}$.

Remark 4.5 Again, the requirement that $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$ and $\Gamma \models_{\text{CL}} \Delta$ cannot provide a *necessary* condition for a classical inference to be K_3^w -valid, even in case Γ is consistent. Indeed, by the definition of K_3^w -consequence, we have $\Gamma \models_{K_3^w} \Delta \cup \Psi$ if $\Gamma \models_{K_3^w} \Delta$, even if $\text{Var}(\Psi) \not\subseteq \text{Var}(\Gamma)$. Just to get a concrete feeling of this, consider that $A, C \models_{K_3^w} A \vee B, C$, and yet the inference may violate $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$: if we take the case where A, B, C are distinct propositional variables p, q, s , respectively, we have that $\text{Var}(\{A, B, C\}) \not\subseteq \text{Var}(\{A, C\})$. As with PWK, however, the unrestricted inclusion requirement helps understand the strategy suggested by Theorem 4.3. Given a classically valid inference from a consistent set Γ to Δ , the theorem suggests to check whether $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$. If this is not the case, the theorem suggests to find a subset $\Delta' \subseteq \Delta$ such that $\Gamma \models_{\text{CL}} \Delta'$ and $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$. If there is at least one such set, then $\Gamma \models_{K_3^w} \Delta$; otherwise, $\Gamma \not\models_{K_3^w} \Delta$.

Another corollary of Theorem 4.3 is:

Corollary 4.6 For every $\Gamma, \Delta \subseteq \Phi_{\mathcal{L}}$, if $\Gamma \not\models_{\text{CL}} \emptyset$, then the following *Local Containment Requirement* holds:

$$(\text{LCR}) \quad \text{If } \Gamma \models_{K_3^w} \Delta \text{ then } \text{Var}(\Delta') \subseteq \text{Var}(\Gamma) \text{ for some } \Delta' \subseteq \Delta \text{ s.t. } \Gamma \models_{\text{CL}} \Delta'$$

Condition LCR from Corollary 4.6 is closely related to the containment requirement that is imposed on consequence by some systems of *containment logic*—see condition $\text{PP}^\equiv$ below. The label ‘local containment requirement’ aims at highlighting this connection—the qualification ‘local’ points at the fact that the inclusion requirement needs not being satisfied by the entire conclusion-set Δ ,

but just by a suitable subset of it. A consequence of Corollary 4.6 is that, if the premise-set Γ is consistent, then K_3^w -consequence looks *analytic* in a sense reminiscent of Kant's: among the concepts or information we can extract from the conclusion, we will always have concepts or information that are already in the premise-set.

4.1 Examples of classical failures

We can exploit the duality from 4.2 in order to individuate failures and validities in K_3^w . In particular, all the classical rules of inference failing in PWK turn valid in K_3^w , including for instance ECQ, CS and the other rules from section 3.1. By contrast, all the valid formulas from PWK turn to be invalid in K_3^w . More in general, K_3^w has no tautology:

Fact 4.1 For every set $\Delta \subseteq \Phi_{\mathcal{L}}$ of formulas, $\emptyset \not\models_{K_3^w} \Delta$. (K_3^w is non-tautological.)

The fact is explained by Theorem 4.3: $\text{Var}(\Delta') \not\subseteq \text{Var}(\emptyset)$ for every non-empty $\Delta' \subseteq \Delta$, to the effect that the inclusion requirements from Theorem 4.3 are not met.¹⁴ A special case of Fact 4.1 is failure of the *Law of Identity* (LI) $\emptyset \models A \supset A$. The duality of PWK and K_3^w also implies that K_3^w is a *paracomplete logic*—that is, $A \vee \neg A$ fails in it—but not a *paraconsistent* one—ECQ is valid in it. Notice that *Disjunctive Adjunction* (DA) fails in K_3^w :

$$A \not\models_{K_3^w} A \vee B \quad \text{DA}$$

Once again, the condition by Theorem 4.3 is not met by this inference.¹⁵ The theorem also explains some interesting valid inferences in K_3^w , such as:

$$\begin{array}{l} A \models_{K_3^w} A, B \\ A \vee B \models_{K_3^w} A \vee \neg A \end{array} \quad \text{DA}'$$

4.2 Discussion of Theorem 4.3 and related results

Theorem 4.3 makes a progress w.r.t. an existing theorem by (Urquhart, 2002), which provides a characterization of single-conclusion K_3^w -consequence. In particular, the result by (Urquhart, 2002) is a special case of our result:

Corollary 4.7 ((Urquhart, 2002), Theorem 2.3.1)

$$\Gamma \models_{K_3^w} B \text{ iff } \begin{cases} \Gamma \models_{\text{CL}} \emptyset & \text{or} \\ \Gamma \models_{\text{CL}} B, \text{ and } \text{Var}(B) \subseteq \text{Var}(\Gamma) \end{cases}$$

which immediately follows from Theorem 4.3 and the assumption that $\Delta = \{B\}$. Again, focus on this limit-case does not allow to distinguish the different contributions made by LCR and the unrestricted containment requirement from Corollary 4.4. Indeed, the two requirements coincide as far as single-conclusion consequence is considered.

¹⁴The failure is also easy to check model-theoretically. For every $\Delta \subseteq \Phi_{\mathcal{L}}$ and $p \in \text{Var}(\Delta)$, there is a $V \in \mathcal{V}_{K_3^w}$ such that $V(p) = \mathbf{n}$. By Fact 2.1, this implies that $V(A) = \mathbf{n}$ for every formula such that $A \in \Delta$. This suffices to dissatisfy Δ .

¹⁵A counterexample to DA is given by any $V \in \mathcal{V}$ such that $V(A) = \mathbf{t}$ and $V(B) = \mathbf{n}$: by Fact 2.1, this implies $V(A \vee B) = \mathbf{n}$.

Comparison of Theorem 4.3 and Corollary 4.7 also shows that, contrary to PWK-consequence, the characterization of multiple/multiple K_3^w -consequence cannot obtain by simply generalizing the characterization of its multiple/single-conclusion case. When it comes to K_3^w -consequence, the generalization from $\text{var}(B) \subseteq \text{var}(\Gamma)$ to $\text{var}(\Delta) \subseteq \text{var}(\Gamma)$ does not give the intended result, as $A \vee B \models_{K_3^w} A \vee B, C$ suffices to show. What we need is to find a *subset* Δ' of Δ that satisfies the variable-inclusion requirement at stake. Notice that our strategy of proof allows us to exploit Theorem 3.4 in order to establish the characterization of multiple/multiple K_3^w -consequence *without* an explicit investigation of what specific generalization of the condition from Corollary 4.7 would do the job.

An interesting consequence of Corollary 4.7, however, is that, when $\Delta = \{B\}$, the condition ' $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$ and $\Gamma \models_{\text{CL}} \Delta$ ' is also a *necessary* condition for Δ to follow from a consistent Γ . Indeed:

Corollary 4.8 For every $\Gamma \subseteq \Phi_{\mathcal{L}}$ and $B \in \Phi_{\mathcal{L}}$, if $\Gamma \not\models_{\text{CL}} \emptyset$, then the following *Global Containment Requirement* holds:

$$(\text{GCR}) \quad \text{If } \Gamma \models_{K_3^w} B, \text{ then } \text{Var}(B) \subseteq \text{Var}(\Gamma)$$

As condition LCR from Corollary 4.6, condition GCR from Corollary 4.8 connects K_3^w to the so-called *containment logic*—see below for further discussion.

Summing up this overview: given a classically valid inference from Γ to Δ with Γ consistent, the variable inclusion requirement LCR from Theorem 4.3 is a *necessary* and *sufficient* condition for Δ to follow from Γ in K_3^w . If we confine to single-conclusion consequence, Corollary 4.7 immediately suggests a simpler requirement.

4.3 K_3^w -consequence and Containment Logic

Containment logic originates from (Parry, 1932). Systems in this family are usually defined by a single-conclusion relation of consequence, and come in two different fashions: some of them impose a containment requirement on valid conditionals, others impose a requirement on the relation $\models$ of logical consequence (or the relation $\vdash$ of derivability).¹⁶ The logic PAI introduced in (Parry, 1932) obeys the so-called *proscriptive principle*, that is a containment requirement of the first kind:

$$\text{PP}^{\rightarrow} \quad \text{If } \models A \rightarrow B, \text{ then } \text{Var}(B) \subseteq \text{Var}(A)$$

Containment systems obeying $\text{PP}^{\rightarrow}$ also include the logics S by (Deutsch, 1977) and S^* by (Daniels, 1986), the logic DAI by (Dunn, 1972) and the logic by (Fine, 1986). These logics—and especially those by (Dunn, 1972) and (Fine, 1986)—are also known as *logics of analytic implication*. The other version of the containment requirement is:

$$\text{PP}^{\models} \quad \text{If } \Gamma \models B, \text{ then } \text{Var}(B) \subseteq \text{Var}(\Gamma)$$

We call *consequence containment logics* the systems that follow this requirement. These include, among others, the logic AC from (Angell, 1977), the first-degree fragment S_{fde} of the logic S by (Deutsch, 1977)— S_{fde} is independently developed under the name AL by (Oller, 1999)—the first-degree entailment S_{fde}^* of S^* developed by (Daniels, 1990)—which is independently developed under

¹⁶For a recent investigation on containment logics, see (Ferguson, 2014a), that provides a computational interpretation of the containment requirement, and (Ferguson, 2014b), that explores containment logics resulting from K_3^w .

the name FDE_ϕ by (Priest, 2010)—the logic RC by (Johnson, 1976), and some of the systems from (Szmuc, 2017) and (Barrio et al., 2016)—more precisely, logics L_{nc} , $L_{nb'e}$, $L_{bb'e}$, $L_{nbb'e}$, as well as logics $L_{\epsilon b'}$ and $L_{b'e}$, which we discuss in Section 5.

Logic K_3^w *vacuously* satisfies $PP^\rightarrow$, since it has no tautologies and, a fortiori, no valid conditional. As for $PP^\models$, the condition is, basically, just GCR. The crucial difference between K_3^w and consequence-containment logic is in the range of application of GCR: in Corollary 4.8, GCR is a *necessary* condition for B to follow from a *consistent* set Γ in K_3^w , while $PP^\models$ makes it a necessary condition for consequence independently from any further condition.

It is clear that K_3^w does not qualify as a containment logic in the sense of $PP^\models$, even if we confine ourselves to single-conclusion consequence. (Ferguson, 2014b) and (Ferguson, 2014a) discuss three strategies to ‘turn’ K_3^w into a containment logic. We refer the reader to those papers for further insights on the issue.

5 Two sublogics of PWK and K_3^w

There is increasing interest nowadays toward *sublogics* of WKLs. For instance, (Barrio et al., 2016) and (Szmuc, 2017) discuss a number of systems where more than one contaminating value is at stake, with all such values being ordered in a *linear hierarchy of contaminating values* (this requires relaxing the notion of contamination, see introduction to Fact 5.1 below). In this section, we show that the methods and results that we have deployed in the previous sections prove helpful also when it comes to (at least some) sublogics of PWK and K_3^w . In particular, we prove that the results from Section 3 and Section 4 extend in a straightforward way to the four-valued logics $L_{\epsilon b'}$ and $L_{b'e}$ from (Barrio et al., 2016) and (Szmuc, 2017).¹⁷ The former obtains by including an additional designated contaminating value to the semantics of K_3^w , the latter obtains by including an additional non-designated contaminating value to the semantics of PWK. Theorem 5.4 below shows that, in a sense, $L_{\epsilon b'}$ is to K_3^w what PWK is to CL: it imposes on PWK the same variable-inclusion requirement that K_3^w imposes on CL. In a similar way, Theorem 5.7 shows that $L_{b'e}$ is to PWK what K_3^w is to CL.¹⁸

In order to interpret $L_{\epsilon b'}$ and $L_{b'e}$, we need to generalize our valuation function V from Definition 2.1 to a valuation function $U : \text{Var} \rightarrow \{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{f}\}$. The valuation extends to arbitrary formulas according to the following definition:

Definition 5.1 (Valuation, 2) A valuation $U : \Phi_{\mathcal{L}} \rightarrow \{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{f}\}$ is the unique extension of a mapping $U : \text{Var} \rightarrow \{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{f}\}$ that is induced by the tables from Table 2

In a nutshell, the tables from Table 2 generalize Table 1 by introducing *two* values $\mathbf{n}_1$ and $\mathbf{n}_2$, where $\mathbf{n}_2$ contaminates all other values (in the sense of contamination supplied by Fact 2.1), and $\mathbf{n}_1$ contaminates all the classical values (in a weaker, yet informally clear, sense of contamination). In order to give a rigorous formal account of this behavior, we need to adjust the notion of contamination from Fact 2.1. First, let us denote by $\mathcal{U}$ be the set $U, U', U'', \dots$ of valuations conforming to Definition 5.1. Once this is done, we notice that the following fact holds for $\mathcal{L}$ when interpreted on Table 2:

¹⁷These logics are called dS_{fde}^w and S_{fde}^w in (Da Re et al., 2018). We keep the notation from (Szmuc, 2017) for the sake of coherence with the labels of the other logics from (Szmuc, 2017) that we mention at the end of Section 4.

¹⁸The names of the two four-valued logics in questions come from a convention that is followed by (Barrio et al., 2016) and (Szmuc, 2017), which denotes by ϵ the non-designated contaminating value from K_3^w and by b' the designated contaminated value from PWK.

Table 2:

	$\neg A$	$A \vee B$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$	$A \wedge B$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$	$A \supset B$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$
$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_1$
$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$	$\mathbf{n}_2$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{t}$

Fact 5.1 (Contamination, 2) For all formulas A in $\Phi_{\mathcal{L}}$, valuation $U \in \mathcal{U}$, and values $\mathbf{n}_i, \mathbf{n}_j \notin \{\mathbf{t}, \mathbf{f}\}$ with $i, j \in N$:

$$U(A) = \mathbf{n}_i \text{ iff } \begin{cases} U(p) = \mathbf{n}_i \text{ for some } p \in \text{Var}(A), & \text{and} \\ U(q) \neq \mathbf{n}_j \text{ for every } p \in \text{Var}(A) \text{ and } j > i \end{cases}$$

In this section, when talking about ‘contaminating values’, we will be referring to values that satisfy Fact 5.1—which gives us, of course, a looser notion of ‘contaminating value’ than the one from Fact 2.1. We believe that it is clear that:

Fact 5.2 For every valuation $U \in \mathcal{U}$ such that $U(p) \neq \mathbf{n}_i$ for $i \in \{1, 2\}$ and for every $p \in \text{Var}$, we can build a corresponding valuation $V \in \mathcal{V}$ such that $V(p) = U(p)$ if $U(p) \in \{\mathbf{t}, \mathbf{f}\}$, and $V(p) = \mathbf{n}$ if $U(p) = \mathbf{n}_j$ for $j \in \{1, 2\}$ and $j \neq i$.

This implies that those valuation for $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ and $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ where just *one* of $\mathbf{n}_1$ and $\mathbf{n}_2$ is assigned are, basically, valuations for PWK and $\mathbf{K}_3^w$. The fact proves helpful in what follows.

Exactly as for PWK and $\mathbf{K}_3^w$, $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ and $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ share the same valuation functions and differ in the way they define their relations of consequence. In particular, $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ -consequence is preservation of $\mathbf{t}$ or $\mathbf{n}_2$ through reasoning:

Definition 5.2 ($\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ -Consequence) $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ -consequence is a relation $\models_{\mathbf{L}_{\mathbf{t}\mathbf{b}'}} \subseteq \wp(\Phi_{\mathcal{L}}) \times \wp(\Phi_{\mathcal{L}})$ such that:

$\Gamma \models_{\mathbf{L}_{\mathbf{t}\mathbf{b}'}} \Delta$ iff For every $U \in \mathcal{U}$, if $U(A) \in \{\mathbf{t}, \mathbf{n}_2\}$ for every $A \in \Gamma$, then $U(B) \in \{\mathbf{t}, \mathbf{n}_2\}$ for some $B \in \Delta$.

By contrast, $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ -consequence is preservation of $\mathbf{t}$ or $\mathbf{n}_1$:

Definition 5.3 ($\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ -Consequence) $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ -consequence is a relation $\models_{\mathbf{L}_{\mathbf{b}'\mathbf{e}}} \subseteq \wp(\Phi_{\mathcal{L}}) \times \wp(\Phi_{\mathcal{L}})$ such that:

$\Gamma \models_{\mathbf{L}_{\mathbf{b}'\mathbf{e}}} \Delta$ iff For every $U \in \mathcal{U}$, if $U(A) \in \{\mathbf{t}, \mathbf{n}_1\}$ for every $A \in \Gamma$, then $U(B) \in \{\mathbf{t}, \mathbf{n}_1\}$ for some $B \in \Delta$.

We adopt, with the necessary adjustments, the same conventions we specified in Section 3.

Both $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ and $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ designate a contaminating value, and both comprise a non-designated contaminating value. The difference is that $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ designates the *most* contaminating value, while $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ designates the *least* contaminating one. Below, we will see that this makes (1) the characterization of $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ similar to that of PWK (in the sense that the same variable inclusion requirement is needed), (2) the characterization of $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ similar to that of $\mathbf{K}_3^w$ (in the same sense).

It is clear by Fact 5.2 that $\mathbf{L}_{\mathbf{t}\mathbf{b}'}$ and $\mathbf{L}_{\mathbf{b}'\mathbf{e}}$ are *sublogics* of both $\mathbf{K}_3^w$ and PWK . In particular, this implies:

Fact 5.3 $\mathcal{L}_{\text{cb}'}$ and $\mathcal{L}_{\text{b'e}}$ are:

1. *non-tautological*: $\emptyset \not\models_S \Delta$ for every $\Delta \subseteq \Phi_{\mathcal{L}}$
2. *paraconsistent*: $\Gamma \not\models_S \emptyset$ for every $\Gamma \subseteq \Phi_{\mathcal{L}}$

where $S \in \{\mathcal{L}_{\text{cb}'}, \mathcal{L}_{\text{b'e}}\}$. A consequence of this is that both logics are paracomplete: $\emptyset \not\models_S A \vee \neg A$. Also, it is clear that $\mathcal{L}_{\text{cb}'}$ and $\mathcal{L}_{\text{b'e}}$ are *monotonic*. As for the relations between $\mathcal{L}_{\text{cb}'}$ and $\mathcal{L}_{\text{b'e}}$, neither is a sublogic of the other. Indeed:

$$\begin{array}{ll} A \wedge \neg A \models_{\mathcal{L}_{\text{cb}'}} A \wedge B & A \wedge \neg A \not\models_{\mathcal{L}_{\text{b'e}}} A \wedge B \\ A \vee B \not\models_{\mathcal{L}_{\text{cb}'}} A \vee \neg A & A \vee B \models_{\mathcal{L}_{\text{b'e}}} A \vee \neg A \end{array}$$

As for $A \wedge \neg A \models A \wedge B$, any valuation $U \in \mathcal{U}$ such that $U(A) = \mathbf{n}_2$ is such that $U(A \wedge \neg A) = \mathbf{n}_2$ and, by Fact 5.1, $U(A \wedge B) = \mathbf{n}_2$. Given Definition 5.2, this suffices to prove $A \wedge \neg A \models_{\mathcal{L}_{\text{cb}'}} A \wedge B$. By contrast, any valuation $U \in \mathcal{U}$ such that $U(A) = \mathbf{n}_1$ and $U(B) = \mathbf{n}_2$ will have $U(A \wedge \neg A) = \mathbf{n}_1$ and, by Fact 5.1, $U(A \wedge B) = \mathbf{n}_2$. Given Definition 5.3, this suffices to prove $A \wedge \neg A \not\models_{\mathcal{L}_{\text{b'e}}} A \wedge B$.

As for $A \vee B \models A \vee \neg A$, any valuation $U \in \mathcal{U}$ such that $U(A) = \mathbf{n}_1$ and $U(B) = \mathbf{n}_2$ is such that $U(A \vee B) = \mathbf{n}_2$, by Fact 5.1, and $U(A \vee \neg A) = \mathbf{n}_1$. Given Definition 5.2, this suffices to prove $A \vee B \models_{\mathcal{L}_{\text{b'e}}} A \vee \neg A$. By contrast, any valuation $U \in \mathcal{U}$ such that $U(A \vee B) = \mathbf{n}_1$ will be such that $U(A) \in \{\mathbf{t}, \mathbf{n}_1, \mathbf{f}\}$, by Fact 5.1. The latter implies that $U(A \vee \neg A) \in \{\mathbf{t}, \mathbf{n}_1\}$. Given Definition 5.3, this suffices to prove $A \vee B \not\models_{\mathcal{L}_{\text{cb}'}} A \vee \neg A$.

5.1 Characterizing logical consequence in $\mathcal{L}_{\text{cb}'}$

With the above notions and facts at hand, we are ready to provide the characterization result for $\mathcal{L}_{\text{cb}'}$:

Theorem 5.4

$$\Gamma \models_{\mathcal{L}_{\text{cb}'}} \Delta \text{ iff } \Gamma' \models_{\mathcal{K}_3^w} \Delta \text{ for at least a non-empty } \Gamma' \subseteq \Gamma \text{ s.t. } \text{Var}(\Gamma') \subseteq \text{Var}(\Delta).$$

Proof. We start with the LTR direction. We first prove that *if* $\Gamma \models_{\mathcal{L}_{\text{cb}'}} \Delta$, *then* $\Gamma' \models_{\mathcal{L}_{\text{cb}'}} \Delta$ for at least a non-empty $\Gamma' \subseteq \Gamma$ such that $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$. Assume the antecedent as the initial hypothesis, and suppose that $\Gamma' \not\models_{\mathcal{L}_{\text{cb}'}} \Delta$ for every $\Gamma' \subseteq \Gamma$ such that $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$. This implies that there is valuation $U \in \mathcal{U}$ such that $U(B) \in \{\mathbf{n}_1, \mathbf{f}\}$ for every $B \in \Delta$ and yet $U(A) \in \{\mathbf{t}, \mathbf{n}_2\}$ for every $A \in \Gamma'$. By Fact 5.1, we have $U(p) \neq \mathbf{n}_2$ for every $p \in \text{var}(\Delta)$, and by this, we have $U(q) \neq \mathbf{n}_2$ for every $q \in \text{var}(\bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma')$. This implies that $U(A) = \mathbf{t}$ for every $A \in \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$. U can be extended to a valuation $U' \in \mathcal{U}$ such that $U'(p) = U(p)$ if $p \in \text{var}(\Delta)$, and $U'(p) = \mathbf{n}_2$ otherwise. This implies that $U'(A) = \mathbf{t}$ for every $A \in \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$, $U'(C) = \mathbf{n}_2$ for every $C \in \Gamma \setminus \bigcup_{\Gamma' \in \mathbf{G}_{\Gamma, \Delta}} \Gamma'$, and $U(B) = \{\mathbf{n}_1, \mathbf{f}\}$ for every $B \in \Delta$. But this in turn contradicts the initial hypothesis, given Definition 5.2. Thus, we have that, *if* $\Gamma \models_{\mathcal{L}_{\text{cb}'}} \Delta$, *then* $\Gamma' \models_{\mathcal{L}_{\text{cb}'}} \Delta$ for at least a non-empty $\Gamma' \subseteq \Gamma$ such that $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$. Since $\mathcal{L}_{\text{cb}'}$ is a sublogic of $\mathcal{K}_3^w$, we conclude that $\Gamma \models_{\mathcal{L}_{\text{cb}'}} \Delta$ implies that $\Gamma' \models_{\mathcal{K}_3^w} \Delta$ for at least a non-empty $\Gamma' \subseteq \Gamma$ such that $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$.

As for the RTL direction, assume as the initial hypothesis that $\Gamma' \models_{\mathcal{K}_3^w} \Delta$ for at least a non-empty $\Gamma' \subseteq \Gamma$ such that $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$. To establish $\Gamma' \models_{\mathcal{L}_{\text{cb}'}} \Delta$, fix any valuation $U \in \mathcal{U}$ such that $U(A) \in \{\mathbf{t}, \mathbf{n}_2\}$ for every $A \in \Gamma'$. Our goal is to show that $U(B) \in \{\mathbf{t}, \mathbf{n}_2\}$ for some $B \in \Delta$. We consider two cases:

Case 1): $U(A) = \mathbf{n}_2$ for some $A \in \Gamma'$. Fix some formula $C \in \Gamma'$ such that $U(C) = \mathbf{n}_2$. By Fact 5.1, there is a $q \in \text{var}(C)$ such that $V(q) = \mathbf{n}_2$. Since $\text{var}(\Gamma') \subseteq \text{var}(\Delta)$ and $C \in \Gamma'$, $q \in \text{var}(\Delta)$. Again by Fact 5.1, this implies $U(B) = \mathbf{n}_2$ for some $B \in \Delta$, as desired.

Case 2): $U(A) \neq \mathbf{n}_2$ for every $A \in \Gamma'$. This implies that $U(A) = \mathbf{t}$ for every $A \in \Gamma'$, and, by Fact 5.1, $U(p) = \mathbf{t}$ for every $p \in \text{var}(\Gamma')$. From this and $\Gamma' \models_{\text{CL}} \Delta$ (which follows from the initial hypothesis $\Gamma' \models_{K_3^w} \Delta$), we have that $U(B) = \mathbf{t}$ for some $B \in \Delta$, as desired.

Since these two cases are jointly exhaustive, we conclude $\Gamma' \models_{L_{\text{cb}'}} \Delta$. From this and the monotonicity of $\models_{L_{\text{cb}'}}$, it follows that $\Gamma \models_{L_{\text{cb}'}} \Delta$. $\blacksquare$

Theorem 5.4 explains $A \vee B \not\models_{L_{\text{cb}'}} A \vee \neg A$. Indeed, although the inference is K_3^w -valid, there is no guarantee that the variables of $A \vee B$ are all contained in those of A —notice that $A \vee B$ is, in turn, the only non-empty subset of $A \vee B$. An immediate corollary of Theorem 5.4 is:

Corollary 5.5 *If $\text{Var}(\Gamma) \subseteq \text{Var}(\Delta)$ and $\Gamma \models_{K_3^w} \Delta$, then $\Gamma \models_{L_{\text{cb}'}} \Delta$.*

That is: *those K_3^w -valid inferences in which the information from the premises is contained in the information from the conclusions are $L_{\text{cb}'}$ -valid inferences.* The corollary explains $A \wedge \neg A \models_{L_{\text{cb}'}} A \wedge B$: the inference is K_3^w -valid (since the logic validates ECQ), and the variables from the premise are contained in the variables from the conclusion. Corollary 5.5 is just a special case of the RTL direction of Theorem 5.4. Also, Theorem 4.3 and Theorem 5.4 together imply the following alternative characterization of $L_{\text{cb}'}$ -consequence:

Corollary 5.6

$$\Gamma \models_{L_{\text{cb}'}} \Delta \quad \text{iff} \quad \Gamma' \models_{\text{CL}} \Delta' \quad \text{for some non-empty } \Gamma' \subseteq \Gamma \text{ and } \Delta' \subseteq \Delta \\ \text{s.t. } \text{Var}(\Delta') \subseteq \text{Var}(\Gamma') \subseteq \text{Var}(\Delta).$$

In a nutshell, Corollary 5.6 explicitly displays the variable-inclusion requirements for K_3^w -consequence, which is implicit in Theorem 5.4 and integrates the requirement for PWK-consequence (which is already explicit in Theorem 5.4, by contrast). This in turn enable us to analyze $L_{\text{cb}'}$ -consequence as a given restriction of CL-consequence. Indeed, Corollary 5.6 implies that the $L_{\text{cb}'}$ -valid inferences are those valid classical inference that satisfy the combination of requirements for K_3^w - and PWK-consequence that is expressed by the nesting $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma') \subseteq \text{Var}(\Delta)$.

5.2 Characterizing logical consequence in $L_{b'e}$

The following is a characterization result for $L_{b'e}$:

Theorem 5.7

$$\Gamma \models_{L_{b'e}} \Delta \quad \text{iff} \quad \Gamma \models_{\text{PWK}} \Delta' \quad \text{for at least a non-empty } \Delta' \subseteq \Delta \text{ s.t. } \text{Var}(\Delta') \subseteq \text{Var}(\Gamma).$$

Proof. Again, we start with the LTR direction. We first prove that *if $\Gamma \models_{L_{b'e}} \Delta$, then $\Gamma' \models_{L_{b'e}} \Delta$ for at least a non-empty $\Delta' \subseteq \Delta$ such that $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$.* Assume the antecedent as the initial hypothesis, and suppose that $\Gamma \not\models_{L_{b'e}} \Delta'$ for every $\Delta' \subseteq \Delta$ such that $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$. This implies that there is valuation $U \in \mathcal{U}$ such that $U(B) \in \{\mathbf{n}_2, \mathbf{f}\}$ for every $B \in \Delta'$ and yet $U(A) \in \{\mathbf{t}, \mathbf{n}_1\}$ for every $A \in \Gamma$. By Fact 5.1 and $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$, this implies $U(B) = \mathbf{f}$ for every $B \in \Delta'$. More in general, we have $U(p) \neq \mathbf{n}_2$ for every $p \in \text{var}(\Gamma)$, and by this, we have $U(q) \neq \mathbf{n}_2$ for every

$q \in \text{var}(\bigcup_{\Delta' \in \mathbf{G}_{\Delta, \Gamma}} \Delta')$. This implies that $U(A) = \{\mathbf{n}_1, \mathbf{t}\}$ for every $A \in \Gamma$. U can be extended to a valuation $U' \in \mathcal{U}$ such that $U'(p) = U(p)$ if $p \in \text{var}(\Gamma)$, and $U'(p) = \mathbf{n}_2$ otherwise. This implies that $U'(A) \in \{\mathbf{t}, \mathbf{n}_1\}$ for every $A \in \Gamma$, $U'(C) = \mathbf{n}_2$ for every $C \in \Delta \setminus \bigcup_{\Delta' \in \mathbf{G}_{\Delta, \Gamma}} \Delta'$, and $U(B) = \mathbf{f}$ for every $B \in \Delta$. But this in turn contradicts the initial hypothesis, given Definition 5.3. Thus, we have that, *if $\Gamma \models_{\mathbf{L}_{b'e}} \Delta$, then $\Gamma \models_{\mathbf{L}_{b'e}} \Delta'$* for at least a non-empty $\Delta' \subseteq \Delta$ such that $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$. Since $\mathbf{L}_{\text{cb'}}$ is a sublogic of PWK, we conclude that $\Gamma \models_{\mathbf{L}_{b'e}} \Delta$ implies $\Gamma \models_{\text{PWK}} \Delta'$ for at least a non-empty $\Delta' \subseteq \Delta$ such that $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$.

As for the RTL direction, assume as the initial hypothesis that $\Gamma \models_{\text{PWK}} \Delta'$ for at least a non-empty $\Delta' \subseteq \Delta$ such that $\text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$. To establish $\Gamma \models_{\mathbf{L}_{\text{cb'}}} \Delta'$, fix any valuation $U \in \mathcal{U}$ such that $U(A) \in \{\mathbf{t}, \mathbf{n}_1\}$ for every $A \in \Gamma'$. Our goal is to show that $U(B) \in \{\mathbf{t}, \mathbf{n}_1\}$ for some $B \in \Delta$. We consider two cases:

Case 1): $U(A) = \mathbf{n}_1$ for some $A \in \Gamma$. Fix some formula $C \in \Gamma$ such that $U(C) = \mathbf{n}_1$. By Fact 5.1, there is a $q \in \text{var}(C)$ such that $U(q) = \mathbf{n}_1$. Remember that $\text{var}(\Delta') \subseteq \text{var}(\Gamma)$, and suppose that $q \in \text{var}(\Gamma) \cap \text{var}(\Delta')$. Since $\text{var}(\Delta') \subseteq \text{var}(\Gamma)$ and $v(p) \neq \mathbf{n}_2$ for every $p \in \text{var}(\Gamma)$, we have $v(q) \neq \mathbf{n}_2$ for every $q \in \text{var}(\Delta')$. Suppose now that $U(A) \in \{\mathbf{t}, \mathbf{n}_1\}$ for every $A \in \Gamma$ and $U(B) = \mathbf{f}$ for every $B \in \Delta'$. By Fact 5.2, this implies that there is a valuation $V \in \mathcal{V}$ such that $V(A) \in \{\mathbf{t}, \mathbf{n}\}$ for every $A \in \Gamma$ and $V(B) = \mathbf{f}$ for every $B \in \Delta'$. But this contradicts the initial hypothesis that $\Gamma \models_{\text{PWK}} \Delta'$.

Case 2): $U(A) \neq \mathbf{n}_1$ for every $A \in \Gamma'$. This implies that $U(A) = \mathbf{t}$ for every $A \in \Gamma$, and, by Fact 5.1, $U(p) = \mathbf{t}$ for every $p \in \text{var}(\Gamma)$. From this and $\Gamma \models_{\text{CL}} \Delta'$ (which follows from the initial hypothesis $\Gamma \models_{\text{PWK}} \Delta'$), we have that $U(B) = \mathbf{t}$ for some $B \in \Delta$, as desired.

Since these two cases are jointly exhaustive, we conclude $\Gamma \models_{\mathbf{L}_{\text{cb'}}} \Delta'$. From this and the Definition of $\models_{\mathbf{L}_{b'e}}$, it follows that $\Gamma \models_{\mathbf{L}_{b'e}} \Delta$. ■

Theorem 5.7 has some interesting consequence, that trace the corollaries of Theorem 4.3. For instance:

Corollary 5.8 *If $\text{Var}(\Delta) \subseteq \text{Var}(\Gamma)$ and $\Gamma \models_{\text{PWK}} \Delta$, then $\Gamma \models_{\mathbf{L}_{b'e}} \Delta$.*

That is: *those PWK-valid inferences in which the information from the conclusions is contained in the information from the premises are $\mathbf{L}_{b'e}$ -valid inferences.* Corollary 5.8 is a special case of the RTL direction of Theorem 5.7. A special cases of the full statement of Theorem 5.7 concerns the single-conclusion version of $\mathbf{L}_{b'e}$ -consequence:

Corollary 5.9

$$\Gamma \models_{\mathbf{L}_{b'e}} B \text{ iff } \Gamma \models_{\text{PWK}} B, \text{ and } \text{Var}(B) \subseteq \text{Var}(\Gamma)$$

which states that the $\mathbf{L}_{b'e}$ -valid single-conclusion inferences are exactly the PWK-valid single-conclusion inferences in which the information from the conclusion is comprised in that from the premises. This in turn explains why the PWK-valid inference from $A \wedge \neg A$ to $A \wedge B$ fails in $\mathbf{L}_{b'e}$, while the PWK-valid inference from $A \vee B$ to $A \vee \neg A$ is valid in $\mathbf{L}_{b'e}$. The former violates the variable-inclusion requirement from Corollary 5.9, the second complies with it. An interesting consequence of Corollary 5.9 is that the single-conclusion version of $\mathbf{L}_{b'e}$ -consequence satisfies the requirement GCR from Section 4. Thus, $\mathbf{L}_{b'e}$ is a *containment logic*.

Finally, Theorem 3.4 and Theorem 5.7 together imply the following alternative characterization of $\mathbf{L}_{b'e}$ -consequence:

Corollary 5.10

$$\Gamma \models_{L_{b'e}} \Delta \quad \text{iff} \quad \Gamma' \models_{CL} \Delta' \quad \text{for some non-empty } \Gamma' \subseteq \Gamma \text{ and } \Delta' \subseteq \Delta \\ \text{s.t. } \text{Var}(\Gamma') \subseteq \text{Var}(\Delta') \subseteq \text{Var}(\Gamma).$$

Again, the corollary unpacks all the variable-inclusion requirements that are implied by Theorem 5.7, while highlighting in particular the variable-inclusion requirements for PWK-consequence, which is just implicit in Theorem 5.7. This in turn enable us to conceive $L_{cb'}$ -consequence as a filter on CL-consequence that is determined by the combination of requirements for PWK- and K_3^w -consequence that is expressed by the nesting $\text{Var}(\Gamma') \subseteq \text{Var}(\Delta') \subseteq \text{Var}(\Gamma)$.

5.3 Discussion of Theorem 5.4 and Theorem 5.7

Sublogics like $L_{cb'}$ and $L_{b'e}$ are attracting increasing attention, and they are natural way to generalize the three-valued contaminating setting from WKLs to more than one contaminating value (see also Section 7 for this). However, very little is known about these logics to this day. The two theorems from the present section make a significant progress in our knowledge of such logics, and we believe that this explains their relevance.

Theorem 5.4 and Corollary 5.5 impose the same variable-inclusion requirements on $L_{cb'}$ -consequence as Theorem 3.4 and Corollary 3.5 impose on PWK-consequence. The last group of results impose those conditions as to restrict classical consequence, the first group impose them as to restrict PWK-consequence. In this sense, we may say that $L_{cb'}$ -consequence is a way to ‘Halldénize’ K_3^w -consequence. Analogously, a look at Theorem 5.7 and Corollary 5.8 suffices to understand that $L_{b'e}$ -consequence is a way to ‘Bochvarize’ PWK-consequence.

The proof of Theorem 5.4 goes along the very same lines as the proof of Theorem 3.4. We believe that, far from being a limit, this shows that our approach in Section 3 is extremely fruitful. In particular, the proof of Theorem 3.4 relies on a mechanism that is given syntactic expression by the variable-inclusion requirement from the theorem. We conjecture that PWK shares this mechanism with every logic that includes a most contaminating designated value in the style of $\mathbf{n}$ in PWK and $\mathbf{n}_2$ in $L_{cb'}$. If our conjecture is right, Theorem 3.4 and Theorem 5.4 could prove the most basic sample of a general characterization method, which would comprise an infinite number of characterization results as its special cases.

Similar considerations go for K_3^w and $L_{b'e}$. Theorem 4.3 and Theorem 5.7 could turn to be the simplest cases of a general characterization method, at least if the variable-inclusion requirement from Theorem 4.3 and Theorem 5.7 is shared, as we conjecture, by every logic that includes a most contaminating non-designated value in the style of $\mathbf{n}_2$ in $L_{b'e}$. Also, notice that we did not derive Theorem 5.7 from a duality result, since we thought in case of two contaminating values interacting, it was important to see the different variable-inclusion requirements directly in action.

Finally, we believe that our results make a significant progress w.r.t. (Ferguson, 2014b, Observation 1), that also provides a clear direction for a general characterization methods for logics endowed with many contaminating values. First, (Ferguson, 2014b, Observation 1) is concerned with single-conclusion consequence relations, while our results can suggest a method that would apply to the more general multiple-conclusion case. Second, and more important, (Ferguson, 2014b, Observation 1) concerns logics where ‘nonsensical values’ are not designated, while Theorem 3.4 and Theorem 5.4 turn to provide an insight that is relevant also for logics that comprise one (or more) designated nonsensical value. Although the insight from (Ferguson, 2014b, Observation 1) easily extends to $L_{b'e}$, it is not clear if it extends naturally to $L_{cb'}$ or, in general, to logics whose

most contaminating value is designated. Thus, we believe that the insights offered by our results are more general than those offered by (Ferguson, 2014b, Observation 1).

We plan to devote future research to the verification of our conjecture and the pursue of a general characterization method for logics endowed with a hierarchy of contaminating value in the style of (Barrio et al., 2016) and (Szmuc, 2017)—see Section 7.

6 Discussion

In this section we briefly review the main philosophical interpretations of Weak Kleene logics, their extension with the so-called *meaningfulness operators* from (Bochvar, 1938) and (Halldén, 1949), and the extension of PWK and K_3^w with detachable conditionals (that is, conditionals obeying MP).

The results that we have presented in Sections 3 and 4 are independent from these issues, but since the philosophical and mathematical literature on Weak Kleene logics (and related systems) is quite sparse, we believe it is worth giving the reader a bird-eye view of the three topics. Also, familiarity with some of the issues below helps understand the open problems that we mention in Section 7.

6.1 Philosophical Interpretation of WKLs

The idea of a *contaminating* non-classical value is recurrent in philosophy, and it comes along with different interpretations. We have seen in Section 3 and Section 4 that (Bochvar, 1938) and (Halldén, 1949) interpret the third value of PWK as *nonsensical*—or *meaningless*, in Bochvar’s jargon.¹⁹ This interpretation goes along with the way **n** propagates to a compound formula from its components: the sense of a compound sentence depends on that of its components, and if some component makes no sense, the sentence as a whole will make no sense either.²⁰

However, the interpretation of PWK as a logic of meaninglessness has been generally met with skepticism. (Brady and Routley, 1973) claim that designating a ‘meaninglessness’ value implies that we can be justified in asserting logical nonsense, which in turn ‘destroys the philosophical point of meaninglessness as a value to be assigned to non-significant sentences’.²¹ The reaction by Brady and Routley presuppose the view that a sentence *A* can be rightfully asserted if and only if it has a designated value (in the model that we take to represent our world); a consequence of this view is that we should select a set *X* of designated values because we (believe that we) can rightfully assert a sentence *A* iff *A* is assigned a value from *X*. Let us call this the ‘assertion-designation harmony’ (ADH).²²

¹⁹(Bochvar, 1938) and (Halldén, 1949) use *nonsensical*—or *meaningless*—as an umbrella term including *paradoxical statements* such as the Liar (Halldén, 1949) and Russell’s paradox (Bochvar, 1938), *vague sentences*, *denotational failure*, *ambiguity* (Halldén, 1949). We do not review them in detail, since this would go much beyond the purpose of the present discussion.

²⁰The principle is also endorsed by (Goddard, 1968) and (Goddard and Routley, 1973). Exactly as (Bochvar, 1938) and (Halldén, 1949), these works are after a logic of meaninglessness.

²¹(Brady, 1976) is also critical of preservation of a ‘meaninglessness’ value. Notice that Brady and Routley seem not to be aware of Halldén’s work. However, their criticism hits the very core of the interpretation of PWK that we are discussing.

²²ADH is a natural adjustment to many-valued settings of the view that the aim of an assertion is to state a *true* sentence—to the effect that a sentence *A* can be rightfully asserted if and only if *A* is true, and to the exclusion of **f** from the range of designated values. This view has been explicitly discussed by (Dummett, 1981; Williams, 1966), where CL is taken as the reasoning framework of reference.

Notice that Brady and Routley are not the only supporters of ADH in the philosophy of many-valued logic. Indeed, ADH seems to be shared also by a major project such as the *dialetheism* by Graham Priest, which is based on the logic LP.²³ In particular, the project in question holds that (true) contradictions may be rightfully asserted—see (Priest, 2018) and (Priest and Sylvan, 1989)—and the assertion of a sentence A fits its own aim iff A is consistently true or *inconsistently* true (Priest, 2006; Priest, 2018). This view goes along with designation of both **t** and **n** in LP—where **n** behaves as per the *strong* Kleene tables exemplified in Section 1.²⁴

Assumption of ADH calls for Brady and Routley’s reaction to the interpretation of PWK as a logic of meaningfulness—that is: the interpretation sounds unconvincing, since it would entitle us to assert any meaningless sentence. Notice that (Halldén, 1949, p. 47) justifies its choice for PWK-consequence by purporting the view that we want to stay as close as possible to classical *validity*, not by defending that nonsense is rightfully assertable.²⁵ Thus, what we describe as the designation of the nonsensical value (alongside truth) would just be a by-product of Halldén’s stance on validity in a three-valued setting, rather than a feature that Halldén took as a conceptual desideratum. However, the (standard) notion of validity is defined in terms of satisfaction, and in a matrix-based logic, this equates with defining the notion in terms of the set of values that a logic designates. Thus, the argument from this particular stance by Halldén’s is not able to counter the reaction displayed by the supporter of ADH: she will keep seeing Halldén’s project as a (dubious) entitlement to assert nonsense. Of course, ADH is not forced upon us: there can be plenty of different views on the conceptual import of designating a set X of value, and on the consequences this would have in our reasoning and assertion practices. However, supporters of PWK as a logic of meaningfulness have not come (at least to our knowledge) with an alternative and articulated view on the conceptual import of designating the ‘meaninglessness’ value alongside truth. Lack of such a view and the popularity of ADH, together, considerably diminish the appeal of the interpretation of PWK as a logic of meaningfulness.²⁶

The logic from (Bochvar, 1938) does not share the problem above, since it does not preserve the third value and, under ADH, this in turn implies that nonsensical sentences are not rightfully assertable. In any case, Bochvar’s third value has been also given other interpretations by (Beall, 2016), (Ferguson, 2014a), and (Fitting, 2006).

(Beall, 2016) advances an interpretation of the value as ‘off-topic’. The main motivating intuition is that any interpreted theory naturally ranges on some possible topics (but not on *all* possible topics); value **n** being assigned to a variable would then ideally represent the variable being off-topic w.r.t. the intended theory; the additional (and in our view, reasonable) assumption that an arbitrary formula is on-topic iff all of its variables are on-topic justifies the contaminating behavior

²³Notice also that some projects in non-classical or many-valued logic cannot get along with ADH. For instance, the semantics for *strict-tolerant consequence* by (Cobreros et al., 2012) does not include a set of designated value (we thank an anonymous reviewer for this remark). Semantics of this kind cannot come together with ADH. Of course, this does not mean that they cannot come with a theory of the connections between assertion and logic. Simply, such a theory will use a package of notions that does not include the notion of a designated value.

²⁴The reading of **n** as ‘inconsistently true’ in LP is justified by the fact that $\mathcal{D}_{LP} = \{\mathbf{n}, \mathbf{t}\}$ and $v(A \wedge \neg A) \neq \mathbf{f}$ iff $v(A) = v(A \wedge \neg A) = \mathbf{n}$, for an arbitrary valuation v defined on the strong Kleene tables. However, notice that alternative interpretations of **n** have been proposed relative to LP—see (Beall and Ripley, 2004) for an example.

²⁵This point is highlighted in (Ferguson, 2014a).

²⁶To be sure, we are not claiming that it is impossible to come with a view that convincingly supports the interpretation of PWK as a logic of meaningfulness. We are simply highlighting what we perceive as limits of the way the interpretation has been proposed thus far.

of ‘being off-topic’, which in turn seems a fit interpretation for **n**.

(Ferguson, 2014a) advances a computational interpretation of the third value as ‘missing variable declaration’, which in turn draws inspiration from the papers on *computation errors* from (McCarthy, 1963) and on *free disjunction* from (Zimmerman, 2000). In this interpretation, value **n** is assigned, ideally, to those formulas that contain a propositional variable p that has not been *declared*—this in turn means that the program has not been instructed to use p as a variable.

Finally, (Fitting, 2006) proposes an epistemic interpretation of K_3^w which equates **t** with ‘all the relevant experts have a positive opinion’ (on the given formula), **f** with ‘all the relevant experts have a negative opinion’, and **n** with ‘some relevant experts have no opinion at all’. This interpretation is sustained by an algebraic reconstruction of the operations from K_3^w in terms of a *cut-down operation* $\oplus$ and its dual $\otimes$ defined on bilattices. We refer to (Ferguson, 2014a) and (Fitting, 2006) for further details.

A third contaminating truth value is also involved in the propositional fragment of Prior’s modal logic **Q** (Prior, 1957; Prior, 1967). Prior deals with the modal problem of referring to contingent objects that do not exist in the actual world, and his main tenet is: sentences that make reference to (at least some) nonexistent entities lack any classical truth value.²⁷ Contamination fits Prior’s focus: ‘Pegasus runs on Fifth Avenue or Bob Dylan is a musician’ makes reference to a nonexistent entity, since ‘Pegasus runs on Fifth Avenue’ does. Thus, the former gets the non-classical value, because the latter does. Interestingly, Prior defines valid inferences as going from *non-false* premises to *non-false* conclusions.²⁸ Thus, the propositional fragment of **Q** would be PWK or a logic closely related to it. It is not clear whether Prior’s proposal is really coherent. On the one hand, he wishes to retain classical tautologies, which prompts preservation of non-false formulas through reasoning. On the other hand, his view on reference to nonexistent entities seems to fit better with the *paracomplete* tradition, and with a reading of the third value as *neither true nor false*. Under this reading, however, preserving the third value through reasoning does not seem to make much sense.²⁹

A contaminating truth value is also involved in the five-valued logic by (Priest, 2010), which is in turn a sublogic of FDE and K_3^w .³⁰ This formalism is introduced to model the logic of the *catuskoti* (tetralemma) from Buddhist philosophy. Its name notwithstanding, in some versions of the catuskoti all the four corners are said not to hold good, and this prompts the need of (at least) a *fifth* option. The four non-contaminating values in (Priest, 2010) are read as *true*, *false*, *both true and false*, *neither true nor false*, as in FDE. The contaminating value is not preserved through reasoning, and it is read by Priest as *ineffable*, on the ground of evidence from Buddhist philosophical literature. The ‘ineffability reading’ seems to fare well with contamination and non-designatedness: a sentence is ineffable if any of its component is, and ineffable sentences should not be asserted or used as premises of our reasoning.

²⁷See (Prior, 1957; Prior, 1967) for the motivations of **Q** and Prior’s view on standard quantified modal logic. (Menzel, 1991) comes with an excellent overview of philosophical background and formal machinery of **Q**.

²⁸Prior seems not to notice that this implies the satisfiability of contradictions, and he is somewhat loose on formal details in (Prior, 1957; Prior, 1967), but evidence for all the above points is convincingly summed up by (Menzel, 1991).

²⁹Notice that preservation of classical tautologies is the only reason given by Prior for preservation of non-falsity, which makes the move look *ad hoc*.

³⁰The same logic is deployed by (Daniels, 1990) and is applied to fiction. The paper does not rise the question of the reading of the contaminating value, though.

6.2 Meaningfulness Operators

The full systems of Bochvar and Halldén extend the language of K_3^w and PWK with *meaningfulness* operators. The operator from (Bochvar, 1938), that we symbolize by $\odot$, is an operator that restitutes *true* formulas when applying to *true* formulas: extending our valuation function V to the new connective, $V(\odot A) = \mathbf{t} \Leftrightarrow V(A) = \mathbf{t}$, and $V(\odot A) = \mathbf{f} \Leftrightarrow V(A) \neq \mathbf{t}$. In the interpretation by Bochvar, the operator reads ‘meaningful and true’. Extension of K_3^w with $\odot$ yields the *external logic* B_3 . The operator from (Halldén, 1949), that we symbolize by $\otimes$, restitutes *true* formulas when applying to *true or false* formulas: $V(\otimes A) = \mathbf{t} \Leftrightarrow V(A) \neq \mathbf{n}$, and $V(\otimes A) = \mathbf{f} \Leftrightarrow V(A) = \mathbf{n}$. In the interpretation by Halldén, the operator reads ‘meaningful’. Extension of PWK with $\otimes$ yields Halldén’s *external logic* H_3 . The following table gives an overview of the formal behavior of the two operators:

Table 3: Operators for ‘meaningful and truth’ and ‘meaningful’

	$\odot A$		$\otimes A$
t	t	t	t
n	f	n	f
f	f	f	t

These operators are also introduced, under the same interpretation, by systems of significance logic—see especially (Goddard, 1968) and (Goddard and Routley, 1973). Also, $\otimes$ is introduced (under a different interpretation) in the logic Q by (Prior, 1957; Prior, 1967), where it is read as ‘it is *statable* that ...’, and this in turn expresses that the formula does not make reference to any nonexistent entities.

The ‘meaningfulness’ interpretation is just one of many possible interpretations. In a paraconsistent three-valued logic, $\otimes$ works as a *consistency operator* in the style of the Logics of Formal Inconsistency (LFIs) by (Carnielli and Marcos, 2002), with $\otimes A$ stating that A does not verify $A \wedge \neg A$. Similarly, $\odot$ works as a *just true operator*, with $\odot A$ stating that A is *consistent and true*—see (Beall, 2009) for a discussion on ‘just true’ devices. In a paracomplete three-valued logic, the two operators receive dual readings: $\otimes$ works as a *bivalence operator*, with $\otimes A$ stating that A is either true or false, and $\odot$ works as a *truth operator*, with $\odot A$ stating that A is true.³¹

6.3 Detachable Conditionals

Conditional $\supset$ from $\mathcal{L}$ fails MP in PWK (Section 3) and the Law of Identity in K_3^w (Section 4). These rules are desirable for (if not constitutive of) conditional.³² Since the two logics are not truth-functionally complete,³³ the failures are easily fixed by adding appropriate primitive conditionals.

³¹If more than three values are into account, the semantics of $\odot$ or $\otimes$ can be extended in different ways, capturing a variety of different notions. See (Omori and Sano, 2013).

³²To be sure, a number of researchers from philosophical logic have argued, on different grounds, that MP is not meaning-constitutive for the conditional. See for instance (Beall, 2013) and (McGee, 1985). However, the opposite view is still the most popular—see for instance (Priest, 2006, p.83): *any conditional worth its salt should satisfy the modus ponens principle*. We do not engage in this debate here, since this goes beyond the purpose of this paper.

³³This is clear by the fact that $\otimes$ cannot be defined in terms of $\neg$, $\vee$, or $\wedge$ in PWK or K_3^w .

Here, we just discuss a pair of options. To PWK, we can add the detachable conditional³⁴ from RM_3 ,³⁵ and, to K_3^w , we can add the conditional from the logic L_3 by Łukasiewicz, that was indeed designed to obey the Law of Identity. We call them $\rightarrow_1$ and $\rightarrow_2$, respectively. Their truth tables are:

Table 4: Conditionals $\rightarrow_1$ from RM_3 and $\rightarrow_2$ from L_3 .

$A \rightarrow_1 B$	t	n	f	$A \rightarrow_2 B$	t	n	f
t	t	f	f	t	t	n	f
n	t	n	f	n	t	t	n
f	t	t	t	f	t	t	t

Whenever B gets value **f** and A gets **t** or **n**, $A \rightarrow_1 B$ gets value **f**. This in turn implies that MP proves valid in an extension of PWK with $\rightarrow_1$. Also, a conditional formula from L_3 gets value **t** if antecedent and consequent has the same value, to the effect that the Law of Identity is valid in an extension of K_3^w with $\rightarrow_2$.

Remark 6.1 No conditional that makes MP or the Law of Identity valid will satisfy *contamination*—a look at Table 4 suffices to realize this. Of course, this rises the question whether we would like to extend contamination to some (all) logical operators beyond $\neg, \vee, \wedge, \supset$, or not. One possible reply is *pragmatic*, and claims that admissibility of connectives that do not satisfy contamination depends on the application one is pursuing. From this point of view, it may be good to have conditionals like $\rightarrow_1$ and $\rightarrow_2$ that simply do the job with fixing MP and Law of Identity. Two replies are *extreme*. The first extreme reply claims that *no* further connective should satisfy contamination. This, however, makes sense just if one has viable conceptual reasons to associate contamination to $\neg, \vee, \wedge, \supset$ and no other connective, and there is hardly any. The second extreme reply claims that *all* further connectives should satisfy contamination; once we accept the principle, this reply would go on, we must accept it with no restriction. The problem with this is that it systematically prevents any attempt to get back *any* of the (desirable) inferential powers that are lost with WKLs. Also, it prevents the introduction of operators $\oplus$ and $\odot$ (defined semantically as per Table 3), to the effect that we would never be able to (truly) say that a sentence is meaningless. A fourth and more reasonable reply has it that *some* further connectives may fail to satisfy contamination, provided that they come with an insightful informal reading. In turn, whether a connectives has an appealing informal meaning depends on the informal interpretation of the whole logic at hand. Thus, for instance, if we interpret K_3^w and PWK as logics of meaninglessness, as Bochvar and Halldén did, then $\rightarrow_1$ and $\rightarrow_2$ will hardly qualify. However, we believe that conditionals $\rightarrow_3$ and $\rightarrow_4$ from Table 5 could qualify:

It is straightforward to check that $\rightarrow_3$ and $\rightarrow_4$ make MP and the Law of Identity valid, respectively ($\rightarrow_4$ also validates MP). Also, their truth tables provide an intuitive informal reading for them.

³⁴Adding a detachable conditional to a paraconsistent logic turns to be troublesome if the language expresses its own truth predicate. In this case, MP opens the way for Curry Paradox. As far as no truth predicate is added, however, a detachable conditional brings no trouble.

³⁵This is a move that has already been suggested for LP in (Priest, 2008). RM_3 is a formalism related to relevant logic, for which a three-valued semantics has also been designed. It owes its name to the criterion of *Relevance* for relevant logic and the so-called ‘Mingle Axiom’ $A \implies (A \implies A)$, where $\implies$ is a so-called *relevant conditional*.

Table 5: Conditionals $\rightarrow_3$ and $\rightarrow_4$.

$A \rightarrow_3 B$	t	n	f	$A \rightarrow_4 B$	t	n	f
t	t	f	f	t	t	f	f
n	f	f	f	n	f	t	f
f	t	f	t	f	t	f	t

A formula like $A \rightarrow_3 B$ reads ‘It is meaningful and true that if A , then B ’—or else, ‘It is just true that if A , then B ’. A formula like $A \rightarrow_4 B$ reads ‘Either it is just true that if A , then B , or both A and B are meaningless’. The two readings can be checked via the definability of $\rightarrow_3$ and $\rightarrow_4$ in extensions of PWK and K_3^w with the operator $\odot$. In particular, $A \rightarrow_3 B = \odot(A \supset B)$. As per $A \rightarrow_4 B$, a bit of trivial formal work shows that the following definition is adequate: $A \rightarrow_4 B = \odot(A \supset B) \vee (\neg \odot(A \vee \neg A) \wedge \neg \odot(B \vee \neg B))$. In turn, this definition helps grasp the reading we propose for $\rightarrow_4$.

6.4 Disjunction in K_3^w and Conjunction in PWK

Disjunction in K_3 . Bochvar’s disjunction is *non-adjunctive*—that is, it fails DA. *Non-adjunctive disjunctions* abound in non-classical logics—some examples include the disjunction from *containment logic*, *multiplicative disjunction* from *linear logic* (Girard, 1987), *intensional disjunction* from *relevant logic* (Anderson and Belnap, 1975), *free-choice disjunction* from (Zimmerman, 2000) and *cut-down disjunction* from (Fitting, 2006). There are a number of reasons to believe that these connectives qualify as disjunction albeit failing DA. In particular, (Anderson and Belnap, 1975), (Girard, 1987), and (Zimmerman, 2000) rely on the fact that disjunction, as practiced in natural language or mathematical reasoning, might not follow the intuition that disjunction *must* make DA valid.

Notice that, with the remarkable exception of a family of *containment logics* and Fitting’s *cut-down disjunction*, all the connectives above are equipped with an intensional semantics, and so one could worry whether some connective can (i) enjoy a *truth-functional semantics*, (ii) *fail DA*, and yet (iii) be considered a ‘*real disjunction*’.³⁶

As for K_3^w -disjunction, the recent (Omori and Szmuc, 2017) provides an interesting contribution. In particular, the paper sheds light on some important conceptual aspect of the issue, and it shows that K_3^w -disjunction cannot be ‘*disjunction, as traditionally conceived*’ (Omori and Szmuc, 2017, p. 281)—that is, it cannot receive a semantics that (a) is truth-functional *and* (b) goes along with the *traditional* truth-functional intuition that a disjunction is true if and only if *at least one* of its disjuncts is true. For this reason, we briefly discuss it here.

(Omori and Szmuc, 2017) interprets K_3^w on a variation of the *plurivalent semantics* by (Priest, 2014). Given a set of truth values $\mathcal{T}$, the semantics by (Priest, 2014) assigns each propositional variable p in the language a set of values in $2^{\mathcal{T}}$. The *negative plurivalent semantics* (nps) by (Omori and Szmuc, 2017) admits the sets from $2^{\{\mathbf{t}, \mathbf{f}\}} \setminus \{\mathbf{t}, \mathbf{f}\}$ as possible (sets of) truth values, and let

³⁶Notice that Fitting’s *cut-down disjunction* is defined as a derived algebraic operation combining different meets and joins. The connective can in turn receive the truth-functional semantics illustrated in Table 1, but the informal interpretation of the connective relies on a way more complex algebraic reading, and the latter does not immediately justify the reading of K_3^w -disjunction as a ‘*real disjunction*’.

$\ddot{D} = \{\ddot{v} \in 2^{\{\mathbf{t}, \mathbf{f}\}} \setminus \{\mathbf{t}, \mathbf{f}\} \mid x \in \{\mathbf{t}\} \text{ for some } x \in \ddot{v}\}$ be the set of designated *sets of values*.³⁷ Thus, a variable p can just receive $\{\mathbf{t}\}$, $\{\mathbf{f}\}$, and $\emptyset$ as a value, and just $\{\mathbf{t}\}$ is to be preserved through reasoning.³⁸ We refer the reader to (Omori and Szmuc, 2017) for the truth clauses for arbitrary formulas from $\mathcal{L}$; here, we are interested to a consequence of the clause of disjunction, that is:

$$\mathbf{t} \in \ddot{v}(A \vee B) \text{ iff for some } x, y \in \{\mathbf{t}, \mathbf{f}\}, x \in \ddot{v}(A), y \in \ddot{v}(B) \text{ and } \mathbf{t} = f^\vee(x, y)$$

where $f^\vee$ is the operation corresponding to $\vee$ in the standard two-element Boolean algebra. It is clear by this equivalence that, if $\ddot{v}(A) = \{\mathbf{t}\}$ and $\ddot{v}(B) = \emptyset$, then $\mathbf{t} \notin \ddot{v}(A \vee B)$, since the clause that $\mathbf{t} \in \ddot{v}(B)$ or $\mathbf{f} \in \ddot{v}(B)$ is not satisfied. This implies that the truth of one disjunct does not suffice to secure the truth of a disjunction.³⁹

This feature is interesting for two reasons. First, (Omori and Szmuc, 2017) shows that, when it comes to the plurivalent semantics currently available, contaminating values can be represented just by means of nps—they cannot be represented, for instance, by the plurivalent semantics introduced in (Priest, 2014). Second, the biconditional above is a *consequence* of the general clause for connectives in the nps from (Omori and Szmuc, 2017, Definition 13), which implies that, for every connective k available, if $\mathbf{t} \in \ddot{v}(k(A, B))$, then $x \in \ddot{v}(A)$ and $x \in \ddot{v}(B)$ for $x \in \{\mathbf{t}, \mathbf{f}\}$. This suffices to realize that, in nps, failure of the traditional account of *disjunction* is just a consequence of a condition that is actually imposed on *all* connectives in $\mathcal{L}$.

In our view, one main contribution by (Omori and Szmuc, 2017) is to clarify the status of K_3^w -disjunction w.r.t. the traditional account of disjunction. In doing this, they help us understand that that arguments (or proofs) to the effect that the K_3^w -disjunction is not traditional disjunction do not amount to arguments that K_3^w -disjunction is not a disjunction in its own right. Indeed, (Omori and Szmuc, 2017) correctly show that K_3^w -disjunction cannot receive a semantics that satisfies both (a) and (b), but this, of course, does not imply that points (i) and (ii) above are incompatible with point (iii).⁴⁰

Coming back to this issue, we believe that a number of readings of K_3^w -disjunction justify the view that the connective is a ‘real’ disjunction. For instance, Bochvar’s reading of the third value $\mathbf{n}$ as *meaningless* does make sense of failure of DA. If we follow that reading, we will be justified in assuming that the connective $\vee$ in K_3^w is good old disjunction, and failure of DA is what happens to disjunction when meaningless expressions are around. In addition, the computational reading by (Ferguson, 2014a) also comes with an intuitive explanation: failure of DA is what happens when one disjunct is true, but not all the variables in the other disjunct have been declared. In our view, these two interpretations justify the reading of $\vee$ in K_3^w as ‘disjunction’, even though it fails DA. In light of this, we believe that the position that (i) and (ii) leads to deny (iii) is not justified, at least when it comes to K_3^w -disjunction.

³⁷The negative plurivalent semantics from (Omori and Szmuc, 2017) is more general than this. However, unless stated otherwise, in this paper we refer just to the special case from (Omori and Szmuc, 2017, Definition 13), which provides the nps interpretation of K_3^w .

³⁸Again, this holds if we consider the nps from (Omori and Szmuc, 2017, Definition 13), but it does not hold for all the special cases of (Omori and Szmuc, 2017, Definition 12).

³⁹The same applies to the more traditional matrix-based semantics for WKLs that we investigate in this paper, in which $v(A \vee B) = \mathbf{t}$ iff $v(A) = \mathbf{t}$ and $v(B) \in \{\mathbf{f}, \mathbf{t}\}$ or $v(A) \in \{\mathbf{f}, \mathbf{t}\}$ and $v(B) = \mathbf{t}$.

⁴⁰Indeed, in order to deny (iii), we would need the additional assumption that a connective that enjoys (a) but not (b) is *no disjunction*, *full stop*.

Conjunction in PWK. Halldén’s conjunction faces a dual failure: the connective is *non-simplifying*—that is, it fails CS (see Section 3). Contrary to failure of DA for disjunction, no tradition in non-classical logic has held that CS is *not* distinctive of conjunction, *when the latter is understood truth-functionally*,⁴¹ and no argument in favor of that has been presented.⁴² Be it as it may, the issue at stake is whether a connective can (i) enjoy a *truth-functional semantics*, (ii) *fail CS*, and yet (iii) qualify as a ‘*real conjunction*’.

One possible argument that CS should *not* be distinctive of conjunction goes as follows: ‘exactly as K_3^w -disjunction, PWK-conjunction is good old conjunction, and failure of CS is what happens to conjunction when meaningless expressions are around’. This would qualify PWK-conjunction as a ‘real conjunction’, albeit enjoying a truth-functional semantics and invalidating CS. However, we see below that this view may turn to be problematic.

(Omori and Szmuc, 2017) also provides an interesting insight on PWK-conjunction. In particular, Omori and Szmuc hold that PWK-conjunction ‘*is conjunction, as traditionally conceived*’, where the traditional account of conjunction holds that ‘a conjunction is true if and only if both its conjuncts are true’—call this TA, for short.⁴³ In order to get an nps interpretation of PWK, take (Omori and Szmuc, 2017, Definition 13) and switch the definition of the designated sets of values to $\ddot{D} = \{\ddot{v} \in 2^{\{\mathbf{t}, \mathbf{f}\}} \setminus \{\mathbf{t}, \mathbf{f}\} \mid x \notin \{\mathbf{f}\} \text{ for all } x \in \ddot{v}\}$ —see (Omori and Szmuc, 2017, pp.278–279) for this. The clauses for the connectives remain the same as the ones from (Omori and Szmuc, 2017, Definition 13), and the nps clause for conjunction from (Omori and Szmuc, 2017) has the following consequence:⁴⁴

$$\mathbf{t} \in \ddot{v}(A \wedge B) \text{ iff for some } x, y \in \{\mathbf{t}, \mathbf{f}\}, x \in \ddot{v}(A), y \in \ddot{v}(B) \text{ and } \mathbf{t} = f^\wedge(x, y)$$

This implies that $\mathbf{t} \in \ddot{v}(A \wedge B)$ if and only if $\{\mathbf{t}\} = \ddot{v}(A) = \ddot{v}(B)$, or: a conjunction is true if and only if all its conjuncts are true. A glimpse at Table 1 suffices to see that PWK-conjunction fits this.

One interesting point of (Omori and Szmuc, 2017) is that one can have a conjunction that satisfies the ‘traditional account of conjunction’ while yet failing CS, as is clear by the nps interpretation of PWK. In our view, one of the reasons that make (Omori and Szmuc, 2017) relevant is that it clarifies that the traditional account of conjunction *does not secure* satisfaction of CS, at least if one follows the (certainly viable) reading of the account by (Omori and Szmuc, 2017). However, we believe that the considerations by (Omori and Szmuc, 2017) could fail to prevent a negative reply to points (i)—(iii). In particular, we believe that, even in considering the clarity brought by the analysis by (Omori and Szmuc, 2017), the supporter of ADH (see beginning of this section) could take PWK-conjunction not to be a ‘real conjunction’. Indeed, under ADH a more appropriate formulation of the ‘traditional account of conjunction’ is: ‘a conjunction is designated if and only if all its conjuncts are designated’. Let us call this the ‘generalized traditional account

⁴¹The *intensional conjunction* or *fusion* from the relevantist tradition (see Section 7) fails CS, but it receives an *intensional semantics*. Beside, its reading as a conjunction is far from being universally assumed. For instance, (Read, 2010) interprets the connective as a *compatibility* operator.

⁴²An exception is (Thompson, 1991). His defense of a non-simplifying conjunction is largely independent from the intensional or extensional nature of the semantics on which the connectives are interpreted. However, we believe that Thompson’s argument connecting non-simplifying conjunction and inductive reasoning are far from being conclusive.

⁴³See (Omori and Szmuc, 2017, p.280). Notice that the arguments from (Omori and Szmuc, 2017) does not specifically deal with PWK-conjunction. However, it is clear from the following considerations that their argument applies to PWK-conjunction in a straightforward way.

⁴⁴We refer the reader to (Omori and Szmuc, 2017) for the exact nps-clause for conjunction.

of conjunction’ (GTA, for short).⁴⁵ We see below why this version should look more appropriate than TA to the supporter of ADH. For the time being, we just notice that GTA is just another way to formulate CS, to the effect that no conjunction failing CS—including PWK-conjunction—can satisfy this version of the traditional account. Thus, GTA prompts a negative reply to points (i)–(iii) above. Just to get a concrete feeling of this in the nps by (Omori and Szmuc, 2017): given the general clause for the connectives, we have that

$$\emptyset = \ddot{v}(A \wedge B) \text{ iff } \emptyset = \ddot{v}(A) \text{ or } \emptyset = \ddot{v}(B)$$

to the effect that, if $\emptyset = \ddot{v}(A)$ and $\{\mathbf{f}\} = \ddot{v}(B)$, then $\emptyset = \ddot{v}(A \wedge B)$. Since $\emptyset$ is a designated set of values in the nps interpretation of PWK, we have that $A \wedge B$ can be designated even in cases where B is false (and hence, non-designated).⁴⁶ Of course, acceptance of GTA has direct consequences also for the view that ‘PWK-conjunction is good old conjunction, and failure of CS is what happens to conjunction when meaningless expressions are around’. Under GTA, PWK-conjunction cannot be ‘good old conjunction’ at all, since in her view, ‘good old conjunction’ is something that satisfies CS. In sum, if we accept GTA as a more appropriate version of the ‘traditional account of conjunction’, then we will give a negative reply to (i)–(iii).

Now we explain why, in our view, GTA would prove more appealing than TA under ADH. Given any matrix-based consequence $\models_{\mathbf{S}}$,⁴⁷ ADH implies a reading of $A \models_{\mathbf{S}} B$ as ‘if one can rightfully assert A , then one can rightfully assert B ’. A key question for the supporters of ADH, then, is $(\star)$ ‘*What can one rightfully assert, in case one can rightfully assert sentences $A_1, \dots, A_n$?*’ If one aims at replying this question, the ‘traditional conception of conjunction’ is better represented by (for every well-formed sentences A and B)

$(\star\star)$ A and B are both rightfully asserted if and only if $A \wedge B$ is rightfully asserted.

Which coincides (under ADH) with GTA and, in turn, with CS.⁴⁸ As we have just seen, GTA comes with a negative reply to (i)–(iii). Of course, the fact that (i)–(iii) are not co-tenable under ADH and GTA does not provide a *conclusive* reason against (i)–(iii) holding together. However, since ADH is a widespread position, and GTA is a convincing way to formulate the traditional account of conjunction, their consequences for (i)–(iii) are a relevant issue that the supporter of PWK-conjunction should explicitly discuss and counter. In absence of a view that replaces ADH and GTA, a positive reply to (i)–(iii) seems hardly likely to pack some punch.⁴⁹

⁴⁵Notice that, as far as the paracomplete $\mathbf{K}_3^w$ is concerned, this formulation coincides with the one by (Omori and Szmuc, 2017), since $\mathcal{D}_{\mathbf{K}_3^w} = \{\mathbf{t}\}$.

⁴⁶The same applies to the more traditional matrix-based semantics for WKLs that we investigate in this paper. Indeed, we have that $v(A \wedge B) = \mathbf{n}$ iff $v(A) = \mathbf{n}$ or $v(B) = \mathbf{n}$. Since $\mathbf{n}$ is a designated value in the semantics for PWK from Section 2, we have that $A \wedge B$ can be designated even in cases where B (say) is false (and hence, non-designated).

⁴⁷Here, we extend the label ‘matrix-based consequence’ as to include the consequence relations defined in nps. This seems reasonable, since nps themselves are based on matrices, with designated value-sets being the non-standard element in these semantics. We extend the term ‘matrix-based logic’ in a similar way.

⁴⁸Notice that, under ADH, $(\star\star)$ collapses on TA just in matrix-based logics *where truth* ($\mathbf{t}$ or $\{\mathbf{t}\}$) *is the only designated value*, like CL, $\mathbf{K}_3$, or $\mathbf{K}_3^w$. With respect to those reasoning frameworks, TA will do the job of the ‘traditional account’ for the supporter of ADH. If other values (or value-sets) such as $\mathbf{n}$ (or $\emptyset$) are designated alongside $\mathbf{t}$ (or $\{\mathbf{t}\}$), however, TA does not do the job anymore for the supporter of ADH, since it does not cover all the cases in which a conjunction is rightfully asserted.

⁴⁹To be sure, ADH does not imply endorsement of $(\star\star)$ —and, hence, of GTA and CS. However, rejection of $(\star\star)$ calls for a justification, at least because no tradition in non-classical logic has supported (i)–(iii). To the best of our

7 Issues for Future Research

This paper makes first steps toward a systematic analysis of WKLs. Here we briefly discuss three open problems that we wish to approach in further developments of our research. First, the analysis of logical consequence in the *external logics* by (Bochvar, 1938) and (Halldén, 1949). Second, the construction of many-valued logics with *two or more* contaminating values. Third, the possibility of a relevantist interpretation of Bochvar’s disjunction and Halldén’s conjunction.

Additionally, one reviewer pointed a further interesting issue at us, namely: ‘*Is it possible to prove the semantical characterization results from Theorem 3.4 and Theorem 4.3 in terms of the sequent calculi by (Coniglio and Corbalan, 2012)?*’ We plan to devote future investigation to this topic, which we take to be a valuable direction of further research on WKLs.

Meaningfulness Operators. (Barrio et al., 2016) is a recent paper that focuses on the external logics from (Bochvar, 1938) and (Halldén, 1949) and their four- and five-valued extensions. In particular, they prove that the extensions of WKLs with $\otimes$ and a transparent truth predicate are not trivial, contrary to the corresponding extensions of their strong kins LP and K_3 . The *Derivability Adjustment Theorems* 7.2.20 and 7.3.22 (Corbalan, 2012) establish how the operators $\odot$ and $\otimes$ help *recapture* the classical tautologies that are lost in K_3^w and the classical inferences that fail in PWK, respectively. In ongoing research, (Ciuni and Carrara, 2018) we provide classical recapture theorems for different families of logics; from these theorems, the recapture results for extensions of K_3^w and PWK with $\otimes$ follows as particular cases. Recapture results for a number of subsystems of the two extensions also follow from the general recapture theorems.

More than one contaminating value. The idea of the contaminating truth value is that it transmits from a subformula to a formula *regardless of the values of the other subformulas*. If we want to have more than one contaminating value, this condition must be relaxed, since *at most* one of the contaminating values will prevail over the other. There are many ways to do this for k contaminating values with $k \geq 2$. One option, which is implicit in (Barrio et al., 2016) and (Szmuc, 2017), is to allow for a (linear) *hierarchy of contaminating values*.⁵⁰ This naturally generates sublogics in the intersection of K_3^w and PWK. The issue of adapting the results from Theorem 3.4 and Theorem 4.3 to this family of many-valued logics is not trivial. In Section 5, we have proved characterization results (Theorems 5.4 and 5.7) for two sublogics of PWK and K_3^w in the style of (Barrio et al., 2016) and (Szmuc, 2017). We plan to devote future research to a general characterization method for many-valued logics based on a linear contamination order.

Relevantist reading of Bochvar’s Disjunction. In section 6, we have seen that failure of DA does not pose a conceptual problem to K_3^w -disjunction, and that, together with Bochvar’s interpretation, the computational interpretation from (Ferguson, 2014a) gives an appealing reading to the failure. Further interpretations are possible, of course. For instance, (Ferguson, 2014a) also explores a *relevantist* interpretation of Bochvar’s disjunction.⁵¹ In particular, the right-introduction

knowledge, a justification of this position has not yet proposed by any project in the philosophy of many-valued logic that comprises ADH, which in turn makes this option hardly appealing.

⁵⁰That is, for every $k \in N$ and $1 \leq i \leq k$, if there are k contaminating values $\mathbf{n}_1, \dots, \mathbf{n}_k$, then value $\mathbf{n}_i$ transmits from A to any formula B where A occurs iff all the other formulas occurring in A have value $\mathbf{t}$, $\mathbf{f}$ or $\mathbf{n}_j$ with $j \leq i$.

⁵¹(Ferguson, 2014a) actually reviews different non-classical traditions that can suggest an interpretations of Bochvar’s disjunction, including *linear logic* (Girard, 1987), and the machineries from (Fitting, 2006) and (Zim-

rule of the non-adjunctive *intensional disjunction* $\oplus$ from relevant logic:

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \oplus B, \Delta}$$

is the same as the introduction rule for Bochvar’s disjunction (Coniglio and Corbalan, 2012). From this, (Ferguson, 2014a) concludes that Bochvar’s disjunction and intensional disjunction may be seen as *the same connective*, in line with the traditional proof-theoretical idea that ‘introductions represent the definitions of the symbols concerned’ (Gentzen, 1969).

As a referee points out, this line of reasoning omits the important fact that the calculus from (Coniglio and Corbalan, 2012) imposes a variable-inclusion requirement $var(A, B) \subseteq var(\Gamma)$ to the right-introduction rule for $\vee$, while the right-introduction rule for $\oplus$ does not come with this restriction. In light of this, we believe that Ferguson’s conclusion is in need for more conclusive support. In our view, the real solidity of a relevantist interpretation can only be decided by a fully developed comparison between the *truth-functional semantics* of Bochvar’s disjunction and the *intensional semantics* of $\oplus$. In future research, we plan to provide a full-fledged relevantist (intensional) semantics for Bochvar’s disjunction. In particular, this semantics would attest the possibility to read Bochvar’s disjunction as a *special kind* of intensional disjunction, which comes with an additional variable-inclusion constraint. This special intensional disjunction would naturally support a special kind of conditional, which very likely combines features typical of analytic implication (Dunn, 1972; Fine, 1986) with features typical of relevant implication (Anderson and Belnap, 1975).

Halldén’s Conjunction. Halldén’s conjunction fails CS (see Section 3), and this is really at odd with our understanding of a conjunction (see Section 6). Again, we believe that comparison with relevant logic could help get a better understanding of what the connective is. As (Ciuni, 2015) notices, Halldén’s conjunction carries some interesting similarities with the *fusion* operator $\circ$ (also called *intensional conjunction*) from relevant logic.⁵² Fusion expresses a notion of *informational compatibility*.⁵³ $A \circ B$ holds at a given information state s iff A and B separately hold at two (possibly different) states s' and s'' that are *compatible* with s , respectively.

Fusion and Halldén’s conjunction share a number of properties, such as *commutativity* and *idempotency*,⁵⁴ which are also properties of the traditional extensional conjunction. In addition, fusion is non-simplifying, exactly as conjunction in PWK—see (Mares, 2004) for this. Again, the similarity stretches further, and in a more significant way: the left-introduction rule for the non-simplifying fusion $\circ$ from relevant logic:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \circ B \vdash \Delta}$$

merman, 2000). In addition, he advances a computational interpretation of Bochvar’s disjunction.

⁵²The same notation is used for fusion and the consistency operator of LFIs, but the two cannot be confused. For one, the consistency operator is unary, while fusion is binary.

⁵³Here we are presupposing the informational interpretation of relevant logic advanced by (Dunn, 1993) and (Dunn and Restall, 2002). Notice that $\circ$ has been first introduced by (Lewis and Langford, 1917) as a *compatibility* operator. (Read, 2010) defends this interpretation of the operator in relevant logic.

⁵⁴Commutativity is expressed by $A \circ B \models B \circ A$ and $A \wedge B \models_{\text{PWK}} B \wedge A$, respectively. Idempotency is expressed by $A \circ A \models A$ and $A \wedge A \models_{\text{PWK}} A$, respectively. Actually, fusion is idempotent only in the systems accepting the ‘mingle’ principle $A \models A \implies A$. The principle is in turn equivalent with $A \circ A \models A$, due to the definition $A \circ B = \neg(A \implies \neg B)$.

is the same as the introduction rule for Halldén’s conjunction (Coniglio and Corbalan, 2012). However, we also have the same kind of mismatch that we had with Bochvar’s and intensional disjunction: again, the calculus from (Coniglio and Corbalan, 2012) imposes a variable-inclusion requirement—in this case, $\text{var}(A, B) \subseteq \text{var}(\Delta)$ —to the left-introduction rule for $\wedge$, and again the left-introduction rule for $\circ$ does not come with such a restriction. In light of these conflicting matches and mismatches, we believe that a semantical approach can help understanding how far we can push the similarities between *Halldén’s conjunction* and *fusion*.

In future research, we plan to provide a full-fledged relevantist (intensional) semantics for Bochvar’s disjunction. In particular, this semantics would attest the possibility to read Bochvar’s disjunction as a *special kind* of intensional disjunction, which imposes an additional variable-inclusion constraint. This special intensional disjunction would naturally support a special kind of relevantist conditional, which we plan to investigate in further research. A confirmation of the fact that Halldén’s conjunction can be read a *special kind* of *fusion* would provide us with a more articulated view on the species of compatibility operators that can obtain in relevant reasoning.

8 Conclusions

In this paper we have established *characterization results* for two Weak Kleene logics dating back to (Bochvar, 1938) and (Halldén, 1949): the *paracomplete* logic K_3^w and the *paraconsistent* logic PWK. These results provide *necessary* and *sufficient* conditions for a set Δ of formulas to be a logical consequence of a set Γ in PWK and K_3^w , respectively. The two results make a significant progress with respect to existing literature, and highlight the role of two particular variable inclusion requirements. Some related theorems by (Coniglio and Corbalan, 2012), (Paoli, 2007) and (Urquhart, 2002) turn to be derivable from our results. Besides, we have generalized the main results of the paper to the four-valued logics $L_{\text{cb'}}$ and $L_{\text{b'e}}$ from (Barrio et al., 2016) and (Szmuc, 2017). The paper also gives an overview of the main philosophical interpretations of Weak Kleene Logics, reviews the so-called *meaningfulness operators* from (Bochvar, 1938) and (Halldén, 1949), and the extensions of Weak Kleene Logics with detachable conditionals. Finally, some issues for future research are briefly discussed.

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